Spectral signature of the pitchfork bifurcation: Liouville equation approach

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The time evolution of probability densities of one-dimensional nonlinear vector fields is studied using a Liouville equation approach. It is shown that the Liouville operator admits a discrete spectrum of eigenvalues of decaying type if the vector field is far from bifurcation. The associated right and left eigenvectors are explicitly constructed for simple models and shown to be distributions rather than regular functions. On the other hand, the spectrum of the Liouville operator may become continuous at the bifurcation point, a phenomenon illustrated explicitly in the paper in the case of the pitchfork bifurcation. The relationship between the spectral decompositions of the Liouville and of the Fokker-Planck equations is discussed. In particular, the spectral decompositions constructed here for the Liouville equation are obtained as the noiseless limit of the well known spectral decompositions of the Fokker-Planck equation of the associated stochastic process.

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I. INTRODUCTION

Complex systems giving rise to chaos or bifurcations display a marked sensitivity to small errors, external disturbances, or internal fluctuations since initial states which are experimentally indistinguishable may follow quite different paths in the course of time. This property highlights the need to complement the traditional pointlike description in terms of trajectories by a probabilistic description, capable of accounting in a natural manner for the increasing delocalization of the dynamics in the system state space.

The statistical approach to deterministic chaos has recently attracted considerable attention. In particular, the eigenvalues and also sometimes the eigenfunctions of the Liouville operator or of its time discretized version (the Frobenius-Perron operator) have been determined for a number of representative models such as one-dimensional maps or scattering type systems [1–6]. The situation is rather different when simple bifurcation phenomena such as pitchfork or Hopf bifurcations are concerned, for which much of the effort has so far concentrated on a stochastic description augmenting the deterministic description by incorporating the effects of fluctuations. One is led this way to a master or a Fokker-Planck equation whose solutions bear in one way or another the signature of the bifurcation under consideration [7–9]. Our objective in the present paper is to develop a probabilistic study of bifurcation phenomena at the level of the Liouville equation in which only the deterministic dynamics is taken into consideration and fluctuations are discarded. As we shall see, this description already brings out clearly the complexity of the bifurcation phenomenon through a radical change of the spectral structure across the bifurcation point. It can therefore be considered as a minimal model illustrating the evolution of probability densities, which is considerably more tractable than the models involving chaotic dynamics, while still displaying a good deal of the complexity usually thought to be limited to chaos. An additional interest is that the study can be carried out for continuous time dynamic systems, which are of paramount interest in nonequilibrium statistical mechanics and for which many of the techniques developed in chaos theory fail.

The general formulation is laid down in Sec. II, where the initial value problem and the spectral representation problem of densities are formulated at the level of the Liouville equation and illustrated on a simple toy model of linear dynamics. In Sec. III the Liouville equation corresponding to the normal form of a pitchfork bifurcation is studied with special emphasis on the change of the spectral properties as the system moves across the bifurcation point. A comparison with a Fokker-Planck equation description is presented in Sec. IV, whereas in Sec. V the main conclusions and suggestions for future studies are summarized.

II. GENERAL FORMULATION

A. Liouville equation and time evolution of statistical ensembles

The evolution of a continuous time dynamic system is given by a set of coupled first-order differential equations of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x};\mu) , \tag{2.1}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{F}$ the vector field, and μ a set of parameters. As is well known [10], Eq. (2.1) induces a Liouville equation for the evolution of the probability density $\rho(\mathbf{x},t)$

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t}=-\nabla\cdot[\mathbf{F}\rho(\mathbf{x},t)]=\hat{L}\rho(\mathbf{x},t)\ , \tag{2.2}$$

where $\hat{L}$ is the Liouville operator. Notice that this equation holds for dissipative as well as for conservative systems, for which Eq. (2.2) reduces to the well known Liouville equation of classical statistical mechanics [11]. In most of the present work we shall be concerned with dissipative systems.

Let us write the formal solution of Eq. (2.1) in the form

$$\mathbf{x}(t)=\mathbf{f}^t(\mathbf{x}_0;\mu)\ , \tag{2.3}$$

where $\mathbf{x}_0$ is the initial condition. One can then verify straightforwardly that the solution of the Liouville Eq. (2.2) reads

$$\rho(\mathbf{x},t)=\int d\mathbf{y}\,\rho_0(\mathbf{y})\delta(\mathbf{x}-\mathbf{f}^t(\mathbf{y};\mu))\ , \tag{2.4}$$

which can also be formally written as $\rho_t=\exp(t\hat{L})\rho_0$. Assuming, in addition, the invertibility of $\mathbf{f}^t$, in agreement with the uniqueness of the solution of Eq. (2.1), we may then write Eq. (2.4) in the alternative form

$$\rho(\mathbf{x},t)=\rho_0[\mathbf{f}^{-t}(\mathbf{x};\mu)]\left|\frac{\partial\mathbf{f}^{-t}(\mathbf{x};\mu)}{\partial\mathbf{x}}\right|\ , \tag{2.5}$$

where the last factor is the absolute value of the Jacobian determinant of the inverse transformation $\mathbf{f}^{-t}$.

B. Spectral decomposition

In general, the Liouville operator and its adjoint $\hat{L}^\dagger=\mathbf{F}\cdot\nabla$ are related by $\hat{L}+\hat{L}^\dagger=-\mathrm{div}\mathbf{F}$. For dissipative systems, the divergence ($\mathrm{div}\mathbf{F}$) is not vanishing so that the Liouville operator $\hat{L}$ is not anti-Hermitian. As a consequence, it is necessary to impose further conditions beyond those usually assumed in classical statistical mechanics in order to discuss the spectral properties of such operators [12]. For the sake of simplicity in the presentation of the general formulation, we restrict our arguments for the moment to the simplest case where $\hat{L}$ has only discrete eigenvalues. We emphasize that a continuous spectrum may also appear, as shown in Sec. III B.

Let $\{s_n\}$ be the eigenvalues of the Liouville operator $\hat{L}$, $\{\phi_n(\mathbf{x})\}$ the corresponding eigenvectors, assumed here for simplicity to span the entire functional space of interest,

$$\hat{L}\phi_n(\mathbf{x})=s_n\phi_n(\mathbf{x})\ . \tag{2.6}$$

As $\{\phi_n(\mathbf{x})\}$ are in general not orthogonal, we also introduce the eigenvalue problem of the adjoint operator $\hat{L}^\dagger$,

$$\hat{L}^\dagger\tilde{\phi}_n(\mathbf{x})=s_n^*\tilde{\phi}_n(\mathbf{x})\ , \tag{2.7}$$

where it has been assumed that s_n is an eigenvalue of finite geometric multiplicity and that both $\hat{L}-s_n$ and $\hat{L}^\dagger-s_n^*$ have closed ranges [13] (see Ref. [14]). Under these conditions ϕ and $\tilde{\phi}$ form a complete and biorthonormal set

$$\langle\tilde{\phi}_m,\phi_n\rangle=\delta_{mn}\ ,$$
$$\sum_{n=0}^{\infty}\phi_n(\mathbf{x})\tilde{\phi}_n^*(\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})\ . \tag{2.8}$$

These relations allow, in principle, for a spectral representation of a probability density obeying the Liouville equation. Indeed, expanding $\rho(\mathbf{x},t)$ in terms of $\phi_n(\mathbf{x})$, using Eq. (2.8),

$$\rho(\mathbf{x},t)=\sum_n c_n(t)\phi_n(\mathbf{x})\ , \tag{2.9}$$

and substituting into Eq. (2.2) one obtains

$$\rho(\mathbf{x},t)=\sum_n c_n(0)e^{s_nt}\phi_n(\mathbf{x})$$
$$=\sum_n\langle\tilde{\phi}_n,\rho_0\rangle e^{s_nt}\phi_n(\mathbf{x})\ , \tag{2.10}$$

where ρ_0 is the initial density.

At this point, it is useful to recall that one of the main roles of the probability density is to evaluate average values of observables such as

$$\langle A\rangle_t=\int A(\mathbf{x})\rho(\mathbf{x},t)d\mathbf{x}$$
$$=\langle A,e^{t\hat{L}}\rho_0\rangle=\langle e^{t\hat{L}^\dagger}A,\rho_0\rangle$$
$$=\sum_{n=0}^{\infty}\langle A,\phi_n\rangle e^{s_nt}\langle\tilde{\phi}_n,\rho_0\rangle\ . \tag{2.11}$$

The last identity in (2.11) can be regarded as another version of the spectral decomposition that is applicable in the case where ϕ_n or $\tilde{\phi}_n$, or both, are distributions in the sense of Schwartz. This is the case for the Liouville equation of the present problem as we shall see in the following. On the other hand, the spectral decomposition (2.10) holds in the sense of functions if ϕ_n are eigenfunctions, but in the sense of distributions if ϕ_n are eigendistributions, according to the treated system. In the latter case, we need to apply both members of the distributionlike identities on test functions $A(\mathbf{x})$, which should be regular enough in order that the quantities $\langle A,\phi_n\rangle$ become well defined real (or complex) numbers and that the series in (2.11) is convergent. A similar procedure should be adopted with respect to the initial densities ρ_0 when the eigenvectors $\tilde{\phi}_n$ of $\hat{L}^\dagger$ are distributions. As a result, the spectral decomposition of the Liouville equation acquires a mathematical meaning only for initial densities ρ_0 and final observables A belonging to appropriate functional spaces of test functions $\mathcal{F}_\rho$ and $\mathcal{F}_A$, where the subscripts ρ and A refer to density and observable spaces.

Depending on the functional space considered, the Liouville operator may have different sets of eigenvalues and eigenvectors and one may wonder whether there exists one particular set constituting a natural spectral representation. We shall attempt to provide an answer to this question by analyzing Eq. (2.11) in conjunction with the solution to the initial value problem Eq. (2.4), in which information on the physically acceptable densities is incorporated at the outset.

It is worth noting that the conditions enunciated at the beginning of this subsection imply the absence of Jordan blocks in the spectral decomposition of $\hat{L}$. A Jordan-block structure arises if the operator $\hat{L}$ admits vectors ϕ satisfying

$$(\hat{L}-s_n\hat{I})^{k-1}\phi\neq 0 \ ,$$

while (2.12)

$$(\hat{L}-s_n\hat{I})^{k}\phi=0$$

for some integer $k>1$ ($\hat{I}$ being the identity operator). Such vectors are called radical vectors of rank k associated with the (degenerate) eigenvalue s_n [15] and are also referred to in the literature on bifurcation theory as generalized eigenvectors [16]. The usual eigenvectors are radical vectors of rank one. The linear subspace formed by all the radical vectors associated with the eigenvalue s_n is an invariant subspace under successive applications of $\hat{L}$, which is called a radical subspace [15]. In the case where not all radical vectors are of rank one, the set of all the eigenvectors is not sufficient to span the whole functional space and other radical vectors need to be introduced in the vector basis. A possible basis can be formed by the radical vectors $\{\phi_{n,k}(\mathbf{x})\}$ satisfying

$$\hat{L}\phi_{n,k}(\mathbf{x})=s_n\phi_{n,k}(\mathbf{x})+\phi_{n,k-1}(\mathbf{x}) \quad \text{with } k=1,2,\ldots,d_n \ , \tag{2.13}$$

where d_n is the dimension of the radical space of s_n and $\phi_{n,0}=0$. A case of degenerate eigenvalue giving rise to Jordan blocks is encountered in Sec. III C.

C. Illustration: A linear one-dimensional vector field

We now illustrate the above formulation on the simple one-variable linear system

$$\frac{dx}{dt}=\mu x \ . \tag{2.14}$$

The Liouville equation reads

$$\frac{\partial\rho}{\partial t}=-\mu\frac{\partial(x\rho)}{\partial x}=\hat{L}\rho \tag{2.15}$$

and the solutions to the initial value problems Eqs. (2.3)–(2.5) are

$$x=f^t(x_0)=x_0e^{\mu t} \tag{2.16}$$

and

$$\rho(x,t)=\int_{-\infty}^{+\infty}\rho_0(y)\delta(x-ye^{\mu t})dy = e^{-\mu t}\rho_0(xe^{-\mu t}) \ . \tag{2.17}$$

To put these expressions in the form of Eq. (2.10), one needs to expand the δ distribution or $\rho_0(xe^{-\mu t})$ in power series, respectively, of $ye^{\mu t}$ and $xe^{-\mu t}$ depending on the sign of μ. One reaches in this way the spectral representation equation

$$\rho(x,t)=\sum_{n=0}^{\infty}e^{\mu n t}\left[\int_{-\infty}^{+\infty}\frac{y^n}{n!}\rho_0(y)dy\right](-1)^n\delta^{(n)}(x) \quad \text{for } \mu<0 \tag{2.18}$$

or

$$\rho(x,t)=\sum_{n=0}^{\infty}e^{-(n+1)\mu t}\times\left[\int_{-\infty}^{+\infty}(-1)^n\delta^{(n)}(y)\rho_0(y)dy\right]\frac{x^n}{n!} \quad \text{for } \mu>0 \ , \tag{2.19}$$

with the notation $\delta^{(n)}(x)=d^n\delta(x)/dx^n$.

The first of these expansions is suitable for $\mu<0$ and applies to initial densities for which the integral with arbitrary powers of y exists and to final observables $A(x)$, which guarantee the convergence of the averages (2.11) obtained by applying the expansion (2.18). It predicts that as $t\to\infty$ all terms but the one for $n=0$ die out, thus yielding $\lim_{t\to\infty}\rho(x,t)=\delta(x)$. This is in agreement with Eq. (2.16). From the standpoint of spectral theory, it suggests that $(-1)^n\delta^{(n)}(x)$ are the eigendistributions of $\hat{L}$, Eq. (2.15), whereas $x^n/n!$ are the eigenfunctions of $\hat{L}^{\dagger}$. The corresponding eigenvalues are given by the rate constants appearing in $\exp(s_nt)$. Therefore, the spectrum is discrete and belongs to the negative real axis, which we summarize, for $\mu<0$, by

$$s_n=n\mu=-n|\mu| \quad \text{with } n=0,1,2,3,\ldots$$
$$\phi_n(x)=(-1)^n\delta^{(n)}(x) \ , \tag{2.20}$$
$$\tilde{\phi}_n(x)=\frac{x^n}{n!} \ .$$

The density-functional space of test functions can be chosen as the set of all infinitely differentiable functions $\rho(x)$ with compact support [17]

$$\mathcal{F}_\rho=\mathcal{C}_0^\infty(\mathbb{R}) \ . \tag{2.21}$$

The support is said to be compact if there exists a constant $k>0$ such that $\rho(x)=0$ for $x<-k$ or $x>k$. In such a case, the left eigenvectors are considered as Schwartz distributions. On the other hand, the observable functional space can be chosen as the space of entire functions [18]

$$\mathcal{F}_A=\mathcal{E} \ . \tag{2.22}$$

This choice is the necessary and sufficient condition so that the series (2.18) is well defined in the sense of distributions once $\mathcal{F}_\rho$ is fixed as in (2.21), as proved in Appendix A.

The second expression Eq. (2.19) is suitable for $\mu>0$ and applies to initial densities that are entire functions. It predicts that as $t\to\infty$ the probability in any finite region of the phase space tends to zero. This is in agreement with Eq. (2.16), which implies that the representative point tends to infinity. From the point of view of spectral theory, it appears that $x^n/n!$ are now the eigenfunctions of $\hat{L}$, whereas $(-1)^n\delta^{(n)}(x)$ are the eigendistributions of $\hat{L}^\dagger$. Notice that despite the escape to infinity in this scattering type system, the eigenvalues remain negative and discrete, for, $\mu > 0$:

$$s_n = -(n+1)\mu \quad \text{with } n = 0,1,2,3,\ldots$$
$$\phi_n(x) = \frac{x^n}{n!} , \qquad (2.23)$$
$$\tilde{\phi}_n(x) = (-1)^n \delta^{(n)}(x) .$$

We note the duality that exists between the spectral decompositions of attracting and repelling fixed points: the eigenvectors of $\hat{L}$ and $\hat{L}^\dagger$ are simply exchanged. As a consequence, the density and observable functional spaces are also exchanged, which corresponds to the operation of time reversal $t \to -t$ mapping the attracting case onto the repelling one.

The above statements on the spectrum of $\hat{L}$ and $\hat{L}^\dagger$ can be verified directly on the explicit form of the Liouville operator, Eq. (2.15). For instance, the eigenvalue problem for $\hat{L}^\dagger$ reads

$$\mu x \frac{d}{dx} \tilde{\phi}(x) = s \tilde{\phi}(x) . \qquad (2.24)$$

One can check straightforwardly that, depending on the choice of μ, this equation admits the polynomials $x^n/n!$ or the δ distributions $(-1)^n \delta^{(n)}(x)$ as eigensolutions, the corresponding eigenvalues being respectively $n\mu$ and $-(n+1)\mu$, in agreement with our earlier conclusions. A similar study can be carried out for $\hat{L}$, except that it is actually more convenient to switch to the Fourier space. Again, a full agreement with the conclusion based on Eqs. (2.18) and (2.19) is obtained.

We remark that the spectrum of the Liouville operator may include eigenvalues other than those given by (2.23) if the functional space is enlarged. For instance, in the case $\mu > 0$, the Liouville operator (2.15) admits the functions x^α with $\mathrm{Re}\alpha > 0$ as eigenfunctions. The corresponding eigenvalue is $s = -\mu(\alpha+1)$, which does not belong to the spectrum (2.23) if α is not an integer. The time evolution (2.17) of initial densities such as $\rho_0(x) = x^\alpha \exp(-x^2)$ $(0 < \mathrm{Re}\alpha < 1)$, which are not analytic at the origin, exhibits a decay

$$\rho(x,t) = e^{-\mu(\alpha+1)t} x^\alpha \exp(-x^2 e^{-2\mu t})$$
$$\simeq e^{-\mu(\alpha+1)t} x^\alpha + O(e^{-\mu(\alpha+3)t}) , \qquad (2.25)$$

which differs from (2.19). If the exponent α satisfies $N \le \mathrm{Re}\alpha \le N+1$, where N is an integer, the initial density $\rho_0(x)$ is N times differentiable, but not $(N+1)$ times. Therefore, $\rho_0(x) \in \mathcal{C}^\nu(\mathbb{R})$ with $\mathrm{Re}\alpha > \nu$, which is the space of N-time differentiable functions with $\rho_0^{(N)}(x)$ of Hölder type at the origin with exponent $\nu - N$, i.e., such that $|\rho_0^{(N)}(x) - \rho_0^{(N)}(0)| \le C|x|^{\nu-N}$ (C being a positive constant). The spectral decomposition (2.19) is then replaced by

$$\rho(x,t) = \sum_{n=0}^{N} e^{-(n+1)\mu t} \rho_0^{(n)}(0) \frac{x^n}{n!} + R(x,t) \qquad (\mu > 0) , \qquad (2.26)$$

where the rest $R(x,t)$ is given by

$$R(x,t) = e^{-\mu t} \int_0^{x e^{-\mu t}} dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{m-1}} dx_m [\rho_0^{(N)}(x_m) - \rho_0^{(N)}(0)] \qquad (2.27)$$

and decreases like

$$|R(x,t)| \le \frac{C|x|^\nu}{\nu(\nu-1)\cdots(\nu-N+1)\exp[-\mu(\nu+1)t]} . \qquad (2.28)$$

The rest cannot be further decomposed into a series, which suggests the existence of a continuous spectrum for $\mathrm{Re}s \le -\mu(\nu+1)$. This is indeed the case, as shown before with the eigenfunction x^α $(\mathrm{Re}\alpha \ge \nu)$ of the Liouville operator. The addition of continuous sets to the Liouvillian spectrum when nonsmooth test functions are allowed is a known property of Pollicott-Ruelle resonances, which has been discussed for Axiom-A systems and expanding maps [2,4,6,19,20]. When we consider nested functional spaces

$$\mathcal{E} \subset \cdots \subset \mathcal{C}^{\nu_m}(\mathbb{R}) \subset \cdots \subset \mathcal{C}^{\nu_2}(\mathbb{R}) \subset \mathcal{C}^{\nu_1}(\mathbb{R}) \qquad (2.29)$$

with

$$\cdots > \nu_m > \cdots > \nu_2 > \nu_1 , \qquad (2.30)$$

the continuous spectrum shrinks and disappears in the limit where the initial densities are taken as entire functions. In this process, we observe that the discrete eigenvalues $s_n = -\mu(n+1)$ progressively emerge as if they were previously hidden by the continuous spectrum which covers a two-dimensional domain of the complex plane s. In this regard, the largest functional space for which the spectrum is discrete (if it exists) provides the unique and robust spectral decomposition of the Liouvillian dynamics, which we describe here.

We note that the preceding discussion for the case $\mu > 0$ and the spectral decomposition (2.19) also concern the case $\mu < 0$ and (2.18). In the case $\mu < 0$, the adjoint of the Liouville operator $\hat{L}^\dagger = \mu x \partial_x$ also admits the functions x^α with $\mathrm{Re}\alpha > 0$ as eigenfunctions with corresponding eigenvalues $s = \mu\alpha$. The functional space of the observables should then be taken as $\mathcal{F}_A = \mathcal{C}^\nu(\mathbb{R})$ with $\mathrm{Re}\alpha > \nu$.

III. LIOUVILLE EQUATION OF A ONE-DIMENSIONAL-VECTOR FIELD AT A PITCHFORK BIFURCATION

One of the simplest nonlinear vector fields describing a bifurcation is given by the one-dimensional (1D) equation

$$\frac{dx}{dt} = \mu x - x^3 . \qquad (3.1)$$

This equation plays a very important role in the description of a variety of far-from-equilibrium phenomena in hydrodynamics, chemical reactions, laser physics, etc., where transitions occur between dynamic regimes characterized by a different number of steady states. For Eq. (3.1), the origin $x=0$ is the only fixed point for $\mu<0$, i.e., before the bifurcation. This fixed point is attracting with a stability exponent $\lambda=\mu$ and we may expect that it behaves in a similar way as in the linear model (2.14) with $\mu<0$. However, this fixed point loses its stability and becomes a repelling fixed point for $\mu>0$. Through the bifurcation, two new fixed points emerge from the origin $x=\pm\sqrt{\mu}$, which take over the lost stability of the trivial solution. The new fixed points are attracting with identical stability exponents $\lambda=-2\mu$.

We emphasize that Eq. (3.1) provides a universal description of the so-called supercritical pitchfork bifurcation where two new steady states emerge in a symmetric manner at the bifurcation. Technically, this equation is known as the normal form of this bifurcation, i.e., it is the vector field which contains the minimal set of terms capturing the essence of the bifurcation. For an analytical vector field, there always exists a nonlinear transformation of the coordinate x reducing the vector field to the form $\dot{x}=\mu x-x^3+O(x^5)$, where the rest of the terms of higher degree than the cubic term controlling the bifurcation can be eliminated by a homeomorphism (i.e., a continuous transformation). We also note that Eq. (3.1) is invariant under a space reflection $x\to -x$. In this regard, this model is often used to discuss far-from-equilibrium transitions involving symmetry breaking. We shall see that this symmetry invariance also finds its expression at the statistical level of the Liouville equation description.

Equation (3.1) can be integrated from some initial position x_0 to obtain

$$x=f^t(x_0;\mu)=x_0\left[\frac{\mu}{x_0^2+(\mu-x_0^2)e^{-2\mu t}}\right]^{1/2}, \tag{3.2}$$

which holds for $\mu\neq 0$ and for all positive times $t>0$. A condition for the validity of this solution is that the quantity under the square root remains positive, which leads to the inequality

$$t>t^*(x_0)=-\frac{1}{2|\mu|}\ln\left[1+\frac{|\mu|}{x_0^2}\right], \tag{3.3}$$

where t^* is a negative time. We may wonder what the origin of this restricted domain of validity is when the flow proceeds backward in time up to $t=t^*(x_0)$ from the initial condition x_0. Actually, the trajectory diverges to infinity at this particular time $t^*(x_0)$ due to the cubic term in the equation of motion (3.1). The divergence is of the type $x(t)\sim(t-t^*)^{-1/2}$. Since we are limited to the forward semigroup for our purposes, we have only to require that the solution is valid for positive times $t>0$, which is always the case.

We also remark that we can reduce the nonlinear equation (3.1) to a linear one by the transformation

$$\zeta=\frac{x^2-\mu}{x^2}\to\dot{\zeta}=-2\mu\zeta\ . \tag{3.4}$$

When $\mu>0$, the attracting fixed points $x=\pm\sqrt{\mu}$ are mapped onto the origin $\zeta=0$, while the repelling fixed point $x=0$ is sent at infinity. On the other hand, the attracting fixed points can be sent at infinity with the alternative linearizing transformation

$$\xi=\frac{x}{\sqrt{\mu-x^2}}\to\dot{\xi}=+\mu\xi\ , \tag{3.5}$$

with $-\sqrt{\mu}<x<+\sqrt{\mu}$.

For the pitchfork bifurcation flow, the Liouville equation reads

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial x}[(\mu x-x^3)\rho] \tag{3.6}$$

and the solution to its initial value problem Eq. (2.4) in the form (2.11) becomes

$$\langle A,e^{t\hat{\mathcal{L}}}\rho_0\rangle=\int_{-\infty}^{+\infty}\rho_0(y)A\left[y\left[\frac{\mu}{y^2+(\mu-y^2)e^{-2\mu t}}\right]^{1/2}\right]dy\ , \tag{3.7}$$

with $y=x_0$. In the following, we shall consider separately the three cases $\mu<0$, $\mu=0$, and $\mu>0$.

A. The subcritical regime $\mu<0$

We start from the expression (3.7), which we rewrite in the form

$$\langle A,e^{t\hat{\mathcal{L}}}\rho_0\rangle=\int_{-\infty}^{+\infty}\rho_0(x)A\left[\frac{|\mu|^{1/2}\xi}{\sqrt{1-\xi^2}}\right]dx\ , \tag{3.8}$$

featuring the new variable

$$\xi=\frac{xe^{-|\mu|t}}{\sqrt{x^2+|\mu|}}\ . \tag{3.9}$$

The main idea is to expand the function A in (3.8) in a Taylor series of $\exp(-|\mu|t)$. The right and left eigenvectors of the Liouville equation $\hat{\mathcal{L}}$ can then be identified in the coefficients of this Taylor series. The right eigenvectors are the objects involving the final observable $A(x)$, while the left eigenvectors are those involving the initial density ρ_0. Expanding A as a function of ξ in a Taylor series, we get

$$A\left[\frac{|\mu|^{1/2}\xi}{\sqrt{1-\xi^2}}\right]=\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\left[\frac{\partial^n}{\partial\xi^n}A\left[\frac{|\mu|^{1/2}\xi}{\sqrt{1-\xi^2}}\right]\bigg|_{\xi=0}\right]. \tag{3.10}$$

Accordingly, we obtain the decomposition

$$\langle A,e^{t\hat{\mathcal{L}}}\rho_0\rangle=\sum_{n=0}^{\infty}e^{-n|\mu|t}\left[\int_{-\infty}^{+\infty}\frac{1}{n!}\left[\frac{x}{\sqrt{x^2+|\mu|}}\right]^n\rho_0(x)dx\right]\times\left[\frac{\partial^n}{\partial\xi^n}A\left[\frac{|\mu|^{1/2}\xi}{\sqrt{1-\xi^2}}\right]\bigg|_{\xi=0}\right], \tag{3.11}$$

from which we read the right and left eigenvectors. In summary, the elements of the spectral decomposition are given, for $\mu<0$, by

$$\begin{aligned} s_n &= n\mu = -n|\mu| \quad \text{with } n=0,1,2,3,\ldots \\ \phi_n(x) &= \frac{\partial^n}{\partial \xi^n}\delta\left[x-\frac{|\mu|^{1/2}\xi}{\sqrt{1-\xi^2}}\right]\Bigg|_{\xi=0}, \\ \tilde{\phi}_n(x) &= \frac{1}{n!}\left[\frac{x}{\sqrt{x^2+|\mu|}}\right]^n, \end{aligned} \tag{3.12}$$

which form a complete basis

$$\sum_{n=0}^{\infty}\phi_n(x)\tilde{\phi}_n(y)=\delta(x-y) \tag{3.13}$$

for test functions belonging to the density-functional space chosen here as the set of integrable functions, while the observable functional space is the set of entire functions

$$\mathcal{F}_\rho=\mathcal{L}^1(\mathbb{R}),\quad \mathcal{F}_A=\mathcal{E}\ . \tag{3.14}$$

The eigenvectors can be rewritten as sums of derivatives of Dirac's distribution. Indeed, we can expand the function $A(x)$ in a Taylor series around the origin $x=0$ and thereafter perform the derivatives over ξ in Eq. (3.11) to obtain

$$\phi_n(x)=\sum_k(-1)^k\delta^{(k)}(x)\times\frac{n!(n-2)!!}{k!(k-2)!!}\frac{\sqrt{|\mu|^k}}{2^{(n-k)/2}\left[\frac{n-k}{2}\right]!} \tag{3.15}$$

with

$$k=\begin{cases}2,4,6,\ldots,n & \text{for } n \text{ even}\\ 1,3,5,\ldots,n & \text{for } n \text{ odd}\end{cases} \tag{3.16}$$

and the convention $1!!=0!!=(-1)!!=1$.

The spectral decomposition (3.12) can be compared with the spectral decomposition (2.20) of the linear model (2.14). Since the fixed point $x=0$ is attracting, the right eigenvectors are distributions, while the left eigenvectors are regular functions. The left eigenvectors $\tilde{\phi}_n$ are regular functions over the whole phase space as for the linear model, which is recovered in the limit where $|\mu|\to\infty$. Near the origin $x=0$, the left eigenvectors behave in a way similar to those of the linear model: $\tilde{\phi}_n=x^n/n!$. However, the cubic nonlinearity of the vector field creates a distortion of the left eigenvectors at large distances where they converge to ±1 depending on the sign of x. On the other hand, the right eigenvectors are given by a sum of derivatives of the Dirac distribution as an effect of the nonlinearity. In the limit $|\mu|\to\infty$, the distribution (3.15) is dominated by the last term with $k=n$ and, consequently, $\phi_n(x)/|\mu|^{n/2}$ approaches the right eigenvectors of the linear model. Pitchfork bifurcation below criticality provides therefore an example of the effect of nonlinearities of the vector field on the spectral decomposition, although the spectrum still contains a single family of discrete eigenvalues.

Other 1D vector fields. The previous example suggests a straightforward generalization to any holomorphic 1D vector field $\dot{x}=F(x)$ containing a single attracting fixed point at the origin, with a stability exponent $\lambda=-|\lambda|$ so that $F(0)=0$ and $F^{(1)}(0)=\lambda\neq0$. According to a theorem by Poincaré [21], there exists a biholomorphic transformation $\xi=g(x)$ such that the flow is linearized in the new coordinate ξ: $\dot{\xi}=\lambda\xi$. The transformation $g(x)$ is analytic inside a disk centered on the origin $x=0$, the radius of which is the smallest distance between the fixed point at origin and the complex fixed points $x_i\in\mathbb{C}$ of $F(x)$: $F(x_i)=0$. If there is no real fixed point other than the origin, nothing prevents the holomorphic extension of the function $g(x)$ to the whole real axis. The flow can then be expressed in terms of the function $g(x)$ and its inverse $g^{-1}(x)$ as

$$f^t(x)=g^{-1}[e^{\lambda t}g(x)]\ . \tag{3.17}$$

Accordingly, we have

$$\begin{aligned}\langle A,e^{t\hat{L}}\rho_0\rangle &= \int_{-\infty}^{+\infty}A[f^t(x)]\rho_0(x)dx\\ &= \int_{-\infty}^{+\infty}A[g^{-1}(e^{\lambda t}\xi)]\frac{\rho_0[g^{-1}(\xi)]}{|g'[g^{-1}(\xi)]|}d\xi\\ &= \sum_{n=0}^{\infty}e^{-n|\lambda|t}\langle A,\phi_n\rangle\langle\tilde{\phi}_n,\rho_0\rangle\ ,\end{aligned} \tag{3.18}$$

with the complete basis

$$\begin{aligned}\phi_n(x) &= \frac{\partial^n}{\partial\xi^n}\delta[x-g^{-1}(\xi)]\Big|_{\xi=0}\ ,\\ \tilde{\phi}_n(x) &= \frac{g(x)^n}{n!}\ .\end{aligned} \tag{3.19}$$

We see that the eigenvalues of the Liouville operator are still given by $s_n=-n|\lambda|$ as in the linear model, the effect of the nonlinearities of the vector field being contained entirely in the right and left eigenvectors.

Multidimensional vector fields. Multidimensional generalizations are also straightforward for holomorphic vector fields (2.1) with a single real-valued fixed point whose stability exponents are satisfying the conditions of the linearization Poincaré theorem [21]. This condition requires that the convex envelope of the d exponents $\{\lambda_1,\ldots,\lambda_d\}$ does not contain the origin in the complex plane of λ. Under this condition, there exists a biholomorphic transformation $\boldsymbol{\xi}=\mathbf{g}(\mathbf{x})$ in a multidisk $\mathcal{D}$ centered at the fixed point (supposed to be $\mathbf{x}=0$) such that the flow is linearized: $\dot{\boldsymbol{\xi}}=\boldsymbol{\Lambda}\cdot\boldsymbol{\xi}$ with the diagonal matrix $\boldsymbol{\Lambda}=\text{diag}(\lambda_1,\ldots,\lambda_d)$. We are thus led to the following theorem.

If all the stability exponents are negative $(\lambda_1,\ldots,\lambda_d<0)$, the Liouville equation (2.2) admits a spectral decomposition in terms of the eigenvalues

$$s_{n_1,\ldots,n_d}=-n_1|\lambda_1|-\cdots-n_d|\lambda_d|\ , \tag{3.20}$$

of the right eigendistributions

$$\phi_{n_1,\ldots,n_d}(\mathbf{x})=\prod_{i=1}^{d}\frac{\partial^{n_i}}{\partial\xi_i^{n_i}}\delta(\mathbf{x}-\mathbf{g}^{-1}(\xi))\Bigg|_{\xi=0}, \tag{3.21}$$

acting on observables $A(\mathbf{x})$ for which $A[\mathbf{g}^{-1}(\xi)]$ is analytic in a ball centered at the origin and containing the domain $\mathbf{g}(\mathcal{D})\cap\mathbb{R}^d$ and of the left eigendistributions

$$\tilde{\phi}_{n_1,\ldots,n_d}(\mathbf{x})=\prod_{i=1}^{d}\frac{1}{n_i!}g_i(\mathbf{x})^{n_i}, \tag{3.22}$$

acting on initial densities $\rho_0(\mathbf{x})$, which are integrable functions in balls $\mathcal{D}\cap\mathbb{R}^d$. The right and left eigendistributions (3.21) and (3.22) form a complete biorthonormal basis on the specified functional spaces.

B. The critical regime $\mu=0$

At the bifurcation point $\mu=0$, the vector field Eq. (3.1) reduces to $\dot{x}=-x^3$, so that the trajectories of the system are now given by

$$x=f^t(x_0)=\frac{x_0}{\sqrt{1+2x_0^2t}}, \tag{3.23}$$

which holds for $t>t^*(x_0)=-1/(2x_0^2)$ and, in particular, for positive times. Contrary to the sub- and supercritical cases, the time evolution is now of algebraic rather than exponential character. At long times the trajectories are slowly attracted to the origin according to the power law $x\sim 1/\sqrt{t}$, a behavior known as *critical slowing down*. We shall see that this critical slowing down has a dramatic effect on the spectrum of the Liouville equation.

The time evolution of an average is given by

$$\begin{aligned}\langle A\rangle_t&=\int_{-\infty}^{+\infty}dx\,\rho_0(x)A\left[\frac{x}{\sqrt{1+2x^2t}}\right]\\&=\int_{-\infty}^{+\infty}dx\,\rho_0(x)\\&\quad\times\left[A(0)+\sum_{n=1}^{\infty}\frac{A^{(n)}(0)}{n!}\frac{x^n}{(1+2x^2t)^{n/2}}\right],\end{aligned} \tag{3.24}$$

where we expanded the observable $A(x)$ in a Taylor series at the origin. Looking for a decomposition of this time evolution in terms of exponential functions of the form $\exp(st)$ with $s<0$, we are led to the conclusion that we must form continuous superpositions of such exponentials in order to reproduce algebraic decays. Indeed, if we use well known properties of the Gamma function [22] we obtain

$$\begin{aligned}\frac{x^n}{(1+2x^2t)^{n/2}}&=\frac{1}{2\Gamma(n/2)}\int_{-\infty}^{0}ds\,e^{st}\left[-\frac{s}{2}\right]^{n/2-1}\\&\quad\times\left[\frac{x}{|x|}\right]^n\exp\left[\frac{s}{2x^2}\right].\end{aligned} \tag{3.25}$$

It follows that the system possesses a continuous spectrum constituted by the entire real negative axis ($\mathrm{Re}\,s<0$, $\mathrm{Im}\,s=0$). The terms in the series of Eq. (3.24) separate according to the parity of n with

$$\left[\frac{x}{|x|}\right]^n=\begin{cases}1 & \text{for } n \text{ even}\\ \mathrm{sgn}(x) & \text{for } n \text{ odd},\end{cases} \tag{3.26}$$

so that the continuous spectrum is doubly degenerate.

The Liouville equation admits therefore the spectral decomposition

$$\begin{aligned}\langle A,e^{t\hat{L}}\rho_0\rangle&=\langle A,\phi_0\rangle\langle\tilde{\phi}_0,\rho_0\rangle\\&\quad+\int_{-\infty}^{0}ds\,e^{st}[\langle A,\phi_{s+}\rangle\langle\tilde{\phi}_{s+},\rho_0\rangle\\&\qquad+\langle A,\phi_{s-}\rangle\langle\tilde{\phi}_{s-},\rho_0\rangle].\end{aligned} \tag{3.27}$$

The eigenvectors associated with the simple eigenvalue $s=0$ are

$$\phi_0(x)=\delta(x),\quad \tilde{\phi}_0(x)=1 \tag{3.28}$$

and correspond to the invariant probability density concentrated at the origin $x=0$. The eigenvectors of even parity of the doubly degenerate continuous spectrum $s\in\mathbb{R}^-$ are

$$\phi_{s+}(x)=\sum_{\substack{n=1\\ n\ \text{even}}}^{\infty}\frac{(-1)^n}{2n!\Gamma(n/2)}\left[-\frac{s}{2}\right]^{n/2-1}\delta^{(n)}(x), \tag{3.29}$$

$$\tilde{\phi}_{s+}(x)=\exp\left[\frac{s}{2x^2}\right]. \tag{3.30}$$

The eigenvectors of odd parity are

$$\phi_{s-}(x)=\sum_{\substack{n=1\\ n\ \text{odd}}}\frac{(-1)^n}{2n!\Gamma(n/2)}\left[-\frac{s}{2}\right]^{n/2-1}\delta^{(n)}(x), \tag{3.31}$$

$$\tilde{\phi}_{s-}(x)=\mathrm{sgn}(x)\exp\left[\frac{s}{2x^2}\right]. \tag{3.32}$$

The spectral decomposition holds for observables A belonging to the functional space $\mathcal{F}_A$ of entire functions of exponential type a with arbitrary $a>0$ [17], i.e., of entire functions $A(z)$ such that

$$\mathrm{Sup}_{z\in\mathbb{C}}|A(z)|\exp(-a|z|)<\infty. \tag{3.33}$$

On the other hand, the functional space of the densities ρ_0 is the set of integrable functions $\mathcal{F}_\rho=\mathcal{L}^1(\mathbb{R})$ so that $\int_0^{\pm\infty}\exp(s/2x^2)\rho_0(x)dx$ exists since $s<0$.

We remark that the left eigenvectors are eigenfunctions of the adjoint Liouville operator

$$\hat{L}^\dagger\tilde{\phi}_{s\pm}=s\tilde{\phi}_{s\pm}, \tag{3.34}$$

with $\hat{L}^\dagger=-x^3\partial_x$. In Appendix B we show how the right eigendistributions $\phi_{s\pm}(x)$ are related to the eigenfunction

$$\chi_s(x)=\frac{1}{x^3}\exp\left[-\frac{s}{2x^2}\right] \tag{3.35}$$

of the Liouville operator $\hat{L}(\cdot)=\partial_x(x^3\cdot)$. This eigenfunction has a defective singularity at the origin $x=0$, which prevents it from being handled as a regular eigenfunction. As a consequence, it must be transformed into the distributions (3.29) and (3.31) to acquire a meaning in the spectral decomposition. We have shown elsewhere [9] that the continuous spectrum arises by accumulation of discrete eigenvalues from both sides of criticality, as shown in Fig. 1. The spectral decomposition we construct here confirms this result. We can further show that the aforementioned accumulation concerns not only the critical spectrum, but also the critical eigenvectors. For instance, the left eigenfunctions (3.30) can be obtained from the subcritical left eigenfunctions (3.12) if the limit $\mu\rightarrow 0$ is taken together with $n\rightarrow\infty$, while keeping $s=\mu n$ constant and n even

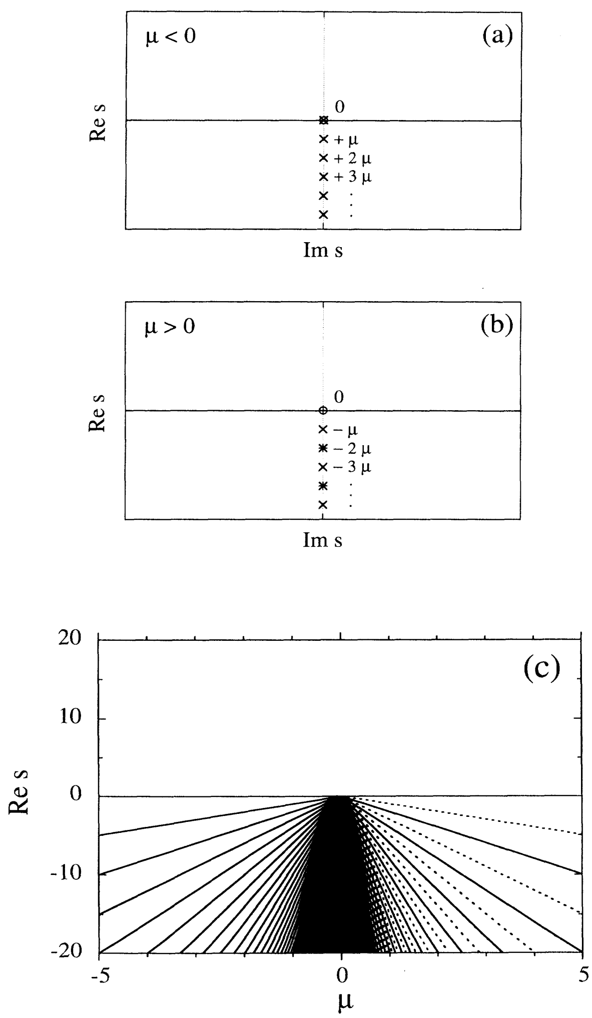

FIG. 1. Location of the eigenvalues of the Liouville operator (3.1) (a) before and (b) after the pitchfork bifurcation at $\mu=0$. (c) Real parts of the eigenvalues as a function of the bifurcation parameter μ through the bifurcation. The imaginary parts are vanishing: $\text{Im}s=0$. The accumulation of eigenvalues near $\mu=0$ generates the continuous spectrum at criticality.

$$\lim_{\mu\rightarrow 0, n\rightarrow\infty, s=\mu n} n!\tilde{\phi}_n(x)=\lim_{n\rightarrow\infty}\left[\frac{x}{\sqrt{x^2+|s|/n}}\right]^n$$
$$=\exp\left[\frac{s}{2x^2}\right]=\tilde{\phi}_{s+}(x)\ . \qquad (3.36)$$

Similar relations hold for the other eigenstates.

As stressed above, the continuous spectrum of the critical Liouvillian dynamics is directly related to the algebraic decay encountered in the critical slowing down. In view of the generality of algebraic behaviors at criticality in far-from-equilibrium bifurcations, we expect that such continuous spectra will occur not only in the case of pitchfork bifurcation, but also in other bifurcations as well.

C. The supercritical regime $\mu>0$

Beyond the pitchfork bifurcation, there exist three fixed points. The origin ($x=0$) is repelling with a stability exponent $\lambda^{(0)}=+\mu$, while the two other fixed points ($x=\pm\sqrt{\mu}$) are attracting with $\lambda^{(\pm)}=-2\mu$. The three fixed points are deeply entangling the Liouvillian dynamics, so that we cannot proceed as in Sec. III A.

We observe that the Liouvillian dynamics keeps the intervals between the fixed points invariant. Accordingly, we should be able to construct the spectral decomposition of the Liouville operator on each one of these intervals separately.

We therefore start by separating the problem in the four intervals we label according to

$$\begin{aligned}&\text{interval } 2+: \ (+\sqrt{\mu},+\infty)\ ,\\ &\text{interval } 1+: \ (0,+\sqrt{\mu})\ ,\\ &\text{interval } 1-: \ (-\sqrt{\mu},0)\ ,\\ &\text{interval } 2-: \ (-\infty,-\sqrt{\mu})\ .\end{aligned} \qquad (3.37)$$

We remark that the intervals $1\pm$ are bordered by the repelling and one of the attracting fixed points. We may therefore expect that the difficulty due to the entangling of the fixed point would appear for these intervals.

We start with the time evolution of an average Eq. (3.7). The integral from $-\infty$ to $+\infty$ is decomposed into four separate integrals in the four intervals (3.37). In a first stage, we derive separately the four corresponding spectral decompositions. Afterwards, the spectral decomposition globally defined on the real axis is obtained by matching the eigenvectors and other radical vectors of the different intervals. Let us begin with the spectral decomposition in the external intervals $2\pm$.

1. The interval 2+

The first term of the integral (3.7) becomes

$$\int_{+\sqrt{\mu}}^{+\infty}dx\ A(x)e^{t\hat{L}}\rho_0(x)=\int_{+\sqrt{\mu}}^{+\infty}dy\ A\left[\frac{\sqrt{\mu}}{\sqrt{1+\xi}}\right]\rho_0(y)\ , \qquad (3.38)$$

where we set

$$\xi=\frac{\mu-y^2}{y^2}\exp(-2\mu t)\ . \tag{3.39}$$

We may now proceed as in Sec. III A to obtain the spectral decomposition

$$\int_{+\sqrt{\mu}}^{+\infty}dx\ A(x)e^{t\hat{L}}\rho_0(x)$$
$$=\sum_{n=0}^{\infty}\exp(-2n\mu t)\langle A,\phi_{2n}^{(2+)}\rangle\langle\tilde{\phi}_{2n}^{(2+)},\rho_0\rangle\ , \tag{3.40}$$

with the eigenvalues

$$s_n=-2n\mu \quad \text{with } n=0,1,2,\ldots \tag{3.41}$$

and the right and left eigendistributions inferred from

$$\langle A,\phi_{2n}^{(2+)}\rangle=\frac{\partial^n}{\partial\xi^n}A\left[\frac{\sqrt{\mu}}{\sqrt{1+\xi}}\right]\Bigg|_{\xi=0}\ , \tag{3.42}$$

$$\langle\tilde{\phi}_{2n}^{(2+)},\rho_0\rangle=\frac{1}{n!}\int_{+\sqrt{\mu}}^{+\infty}\left[\frac{\mu-x^2}{x^2}\right]^n\rho_0(x)dx\ . \tag{3.43}$$

Since $|(\mu-x^2)/x^2|<1$ in the interval 2+, the expansion (3.40) holds for any entire observables $A(x)$ and absolutely integrable densities $\rho_0(x)$.

2. *The interval 2−*

For this interval, we proceed similarly as for 2+. We obtain

$$\int_{-\infty}^{-\sqrt{\mu}}dx\ A(x)e^{t\hat{L}}\rho_0(x)$$
$$=\sum_{n=0}^{\infty}\exp(-2n\mu t)\langle A,\phi_{2n}^{(2-)}\rangle\langle\tilde{\phi}_{2n}^{(2-)},\rho_0\rangle\ , \tag{3.44}$$

with the same eigenvalues (3.40), but the right and left eigendistributions

$$\langle A,\phi_{2n}^{(2-)}\rangle=\frac{\partial^n}{\partial\xi^n}A\left[-\frac{\sqrt{\mu}}{\sqrt{1+\xi}}\right]\Bigg|_{\xi=0}\ , \tag{3.45}$$

$$\langle\tilde{\phi}_{2n}^{(2-)},\rho_0\rangle=\frac{1}{n!}\int_{-\infty}^{-\sqrt{\mu}}\left[\frac{\mu-x^2}{x^2}\right]^n\rho_0(x)dx\ , \tag{3.46}$$

which hold under the same conditions as in the interval 2+.

3. *The interval 1+*

As we mentioned earlier, this interval $(0,+\sqrt{\mu})$ is limited by two fixed points. This complicates the problem, since one fixed point is repelling and should induce a spectral decomposition of the type (2.19), whereas the second fixed point is attracting and should rather induce a decomposition of the type (2.18). Since these decompositions have opposite behaviors at the level of the right and left eigenvectors, we expect an additional difficulty to emerge.

To overcome this difficulty, we proceed by a recursive construction of the eigenvectors and other radical vectors spanning the invariant subspace associated with each eigenvalue. At each step of the construction, we look for the dominant term of the asymptotic time evolution of the average $\langle A\rangle_t$, i.e., the term with the slowest decay which allows us to identify the corresponding radical vectors. Then, we subtract this term from the average and we look for the subdominant term, which is the next slowest decay term. The invariant subspace associated with the next eigenvalue is identified in this way and we continue recursively the construction for still faster decaying modes.

In the construction, we need to go back and forth from the position x of the trajectory (3.2) at time t to the position x_0 at time $t=0$. Indeed, the use of the position x allows us to control the local behavior of the observable $A(x)$ near $x=+\sqrt{\mu}$ at asymptotic times $t\to+\infty$ when members of the statistical ensemble accumulate at the attracting fixed point $x=+\sqrt{\mu}$. On the other hand, we need to use the initial position x_0 to control the behavior of the initial density $\rho_0(x_0)$ near the repeller $x=0$ since its vicinity is depleted of particles as $t\to+\infty$. This duality was already at the basis of the two expansions (2.18) and (2.19). The starting point is to express the time evolution equation (3.7) in both variables x and $x_0=y$, which are related by (3.2)

$$\langle A\rangle_t|_{(0,\sqrt{\mu})}=\int_0^{+\sqrt{\mu}}dy\ A\left[\frac{y\sqrt{\mu}}{\sqrt{y^2+(\mu-y^2)\exp(-2\mu t)}}\right]\rho_0(y) \tag{3.47}$$

$$=\int_0^{+\sqrt{\mu}}dx\frac{\mu^{3/2}\exp(-\mu t)}{[\mu-x^2+x^2\exp(-2\mu t)]^{3/2}}A(x)\rho_0\left[\frac{x\sqrt{\mu}\exp(-\mu t)}{\sqrt{\mu-x^2+x^2\exp(-2\mu t)}}\right]\ . \tag{3.48}$$

The change of variables connecting Eqs. (3.47) and (3.48) will be used repetitively in the construction, the details of which are given in Appendix C.

From Eq. (3.48), we obtain the spectral decomposition

$$\begin{aligned}\langle A\rangle_t|_{(0,\sqrt{\mu})}=&\langle A,\phi_0^{(1+)}\rangle\langle\tilde{\phi}_0^{(1+)},\rho_0\rangle+\exp(-\mu t)\langle A,\phi_1^{(1+)}\rangle\langle\tilde{\phi}_1^{(1+)},\rho_0\rangle\\&+\exp(-2\mu t)(\langle A,\phi_{2a}^{(1+)}\rangle\langle\tilde{\phi}_{2a}^{(1+)},\rho_0\rangle+\langle A,\phi_{2b}^{(1+)}\rangle\langle\tilde{\phi}_{2b}^{(1+)},\rho_0\rangle\\&-\mu^2t\langle A,\phi_{2b}^{(1+)}\rangle\langle\tilde{\phi}_{2a}^{(1+)},\rho_0\rangle)+O(\exp(-3\mu t))\ .\end{aligned} \tag{3.49}$$

The three first invariant subspaces of the spectral decomposition are the following. For the eigenvalue $s_0=0$,

$$\langle A,\phi_0^{(1+)}\rangle = A(\sqrt{\mu}) , \tag{3.50}$$

$$\langle \tilde{\phi}_0^{(1+)},\rho_0\rangle = \int_0^{\sqrt{\mu}} dy\, \rho_0(y) ; \tag{3.51}$$

for the eigenvalue $s_1=-\mu$,

$$\langle A,\phi_1^{(1+)}\rangle = \int_0^{\sqrt{\mu}} dx \frac{\mu^{3/2}}{(\mu-x^2)^{3/2}} [A(x)-A(\sqrt{\mu})] , \tag{3.52}$$

$$\langle \tilde{\phi}_1^{(1+)},\rho_0\rangle = \rho_0(0) ; \tag{3.53}$$

for the eigenvalue $s_2=-2\mu$,

$$\langle A,\phi_{2s}^{(1+)}\rangle = \int_0^{\sqrt{\mu}} dx \frac{\mu^2 x}{(\mu-x^2)^2} \times \left[A(x)-A(\sqrt{\mu}) - \frac{x^2-\mu}{2\sqrt{\mu}} A'(\sqrt{\mu}) \right] , \tag{3.54}$$

$$\langle \tilde{\phi}_{2a}^{(1+)},\rho_0\rangle = \rho_0'(0) , \tag{3.55}$$

$$\langle A,\phi_{2b}^{(1+)}\rangle = \frac{\sqrt{\mu}}{2} A'(\sqrt{\mu}) , \tag{3.56}$$

$$\langle \tilde{\phi}_{2b}^{(1+)},\rho_0\rangle = \int_0^{\sqrt{\mu}} dy \frac{y^2-\mu}{y^2} \times \left[\rho_0(y) - \left[\frac{\mu}{\mu-y^2}\right]^{3/2} \rho_0(0) - \frac{\mu y}{\mu-y^2}\rho_0'(0) \right] . \tag{3.57}$$

Applying these distributions on test functions, we can show that they obey

$$\hat{L}\phi_0^{(1+)}=0, \quad \hat{L}^\dagger\tilde{\phi}_0^{(1+)}=0 , \tag{3.58}$$

$$\hat{L}\phi_1^{(1+)}=-\mu\phi_1^{(1+)}, \quad \hat{L}^\dagger\tilde{\phi}_1^{(1+)}=-\mu\tilde{\phi}_1^{(1+)} , \tag{3.59}$$

$$\hat{L}\begin{bmatrix}\phi_{2a}^{(1+)}\\ \phi_{2b}^{(1+)}\end{bmatrix} = \begin{bmatrix}-2\mu & -\mu^2\\ 0 & -2\mu\end{bmatrix}\begin{bmatrix}\phi_{2a}^{(1+)}\\ \phi_{2b}^{(1+)}\end{bmatrix} , \tag{3.60}$$

$$\hat{L}^\dagger\begin{bmatrix}\tilde{\phi}_{2a}^{(1+)}\\ \tilde{\phi}_{2b}^{(1+)}\end{bmatrix} = \begin{bmatrix}-2\mu & 0\\ -\mu^2 & -2\mu\end{bmatrix}\begin{bmatrix}\tilde{\phi}_{2a}^{(1+)}\\ \tilde{\phi}_{2b}^{(1+)}\end{bmatrix} . \tag{3.61}$$

Equation (3.60) shows that the Liouville operator acting on the subspace spanned by $\phi_{2a}^{(1+)}$ and $\phi_{2b}^{(1+)}$ is not diagonalizable and corresponds to a Jordan block. As a consequence of the appearance of Jordan blocks, the decay dynamics of the time evolution is slower than exponential, as reflected by the presence of terms of the form $t^m \exp(-n\mu t)$ in the expansion (3.49). Because the distributions are well behaved at the fixed points, the spectral decomposition (3.49) holds for analytic functions $A(x)$ and $\rho_0(x)$ without extra restriction on the behavior of these functions at the fixed points.

4. *The interval 1−*

A spectral decomposition similar to (3.49)–(3.61) holds in the interval 1−, which is the symmetric of the interval 1+ by a reflection $x\rightarrow -x$. Therefore, the distributions defining the eigenvectors and other radical vectors are similar, except that $\sqrt{\mu}$ has to be replaced by $-\sqrt{\mu}$ and the integrals from 0 to $+\sqrt{\mu}$ by integrals from $-\sqrt{\mu}$ to 0.

5. *Comparison with the spectrum obtained by the trace formula*

In Ref. [9], we showed that the spectrum of the Liouville operator associated with a general 1D vector field can be obtained from the stability exponents of the linearized dynamics in the vicinity of each fixed point. Moreover, the spectrum is the set of all the eigenvalues of all the fixed points. In the present case, this result implies that the spectrum is composed of the doubly degenerate eigenvalues

$$s_n^{(\pm)} = -2\mu n , \tag{3.62}$$

associated with the attracting fixed points $x=\pm\sqrt{\mu}$, together with the simply degenerate eigenvalues

$$s_n^{(0)} = -\mu(n+1) , \tag{3.63}$$

associated with the repelling fixed point $x=0$ (with $n\in\mathbb{N}$). According to Eqs. (3.62) and (3.63), the dimensions of the invariant subspaces corresponding to $s=0$, $-\mu$, and -2μ are, respectively, 2, 1, and 3.

The systematic theory developed herein allowed us to recover this result and, furthermore, to achieve the explicit construction of the associated radical vectors. Actually, the radical vectors of the different intervals (3.37) must be matched together at the fixed points in order to construct radical vectors which are globally defined on the real axis. Let us first consider the eigenvalue $s=0$, which is common to the four intervals. *A priori,* we have four different eigenvectors in the four different intervals. Therefore, we have here an eigenspace of apparent dimension 4 obtained by linear combinations of these eigenvectors with four independent parameters. However, comparing (3.42), (3.45), and (3.50), we see that the right eigenvectors $\phi_1^{(\alpha)}$ must be matched together at the points $x=\pm\sqrt{\mu}$ in order to maintain the consistency of the spectral decomposition. The matching at two points therefore reduces by two the number of independent parameters so that the eigenspace is of dimension 2, as expected.

On the other hand, the eigenvalue $s=-\mu$ is concerned with the escape dynamics from the repelling fixed point $x=0$ and appears only in the internal intervals 1± so that the apparent dimension of the associated eigenspace is equal to 2. Comparing (3.43) and (3.46) with (3.53), we observe that this eigenvector is pointlike at $x=0$. A matching must be carried out at this fixed point, which introduces one constraint so that the dimension of the associated eigenspace is equal to 1, as expected.

The situation is more complicated for the eigenvalue

$s=-2\mu$, but the reasoning is strictly the same. Here the invariant (radical) subspace is apparently of dimension 6 since each of the external intervals $2\pm$ contributes by one dimension and each of the internal intervals $1\pm$ by two dimensions. In (3.54)–(3.57), we observe that $\tilde{\phi}_{2a}^{(1+)}$ is pointlike at $x=0$, while $\phi_{2b}^{(1+)}$ is pointlike at $x=+\sqrt{\mu}$. Symmetric pointlike radical vectors are found in the symmetric interval $1-$. As a consequence, we have to carry out a matching three times at the three fixed points. (We emphasize here that the matching concerns the distributions which are pointlike.) The dimension of the radical subspace is therefore reduced from 6 to 3, again as expected from (3.62) and (3.63). In conclusion, the present theory is in full agreement with the trace formula result [9].

A final remark is in order about the Jordan blocks appearing in the spectral decomposition and the related time dependence in $t^m \exp(-n\mu t)$. The origin of this phenomenon lies in the degeneracy of the stability exponents of the repelling and attracting fixed points. Since the stability exponents are in the ratio $\lambda^{(\pm)}=-2\lambda^{(0)}$, some of the eigenvalues (3.62) and (3.63) of the Liouville operator turn out to be degenerate, which is a necessary condition for the existence of Jordan blocks in the spectral decomposition of the Liouville operator. The calculation in Appendix C proves that this is indeed the case. For a more general vector field, such degeneracies should not exist. As a consequence, Jordan block would not appear and usual exponential decay would hold.

IV. COMPARISON BETWEEN THE LIOUVILLE AND FOKKER-PLANCK EQUATIONS

As mentioned in the Introduction, a probabilistic approach to bifurcations has been developed in the past at the mesoscopic level of description [7–9]. The starting point is to augment the deterministic description Eq. (2.1) by a *random force* accounting for the effect of fluctuations or disturbances of external origin. One obtains in this way a stochastic differential equation known as the Langevin equation

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x};\mu)+\mathbf{W} \ , \tag{4.1}$$

in which the random force $\mathbf{W}$ is usually modeled as an isotropic Gaussian white noise

$$\langle \mathbf{W}(t)\rangle=0, \quad \langle \mathbf{W}(t)\mathbf{W}(t')\rangle=2D\mathsf{I}\delta(t-t') \ , \tag{4.2}$$

where I is the identity matrix. It is well known from probability theory that Eqs. (4.1) and (4.2) define a diffusion process, whose probability density ρ satisfies the Fokker-Planck equation

$$\partial_t\rho+\nabla\cdot(\mathbf{F}\rho)=D\Delta\rho \ . \tag{4.3}$$

Fokker-Planck equations such as (4.3) are known to admit standard spectral decompositions in terms of the eigenvalues and eigenvectors of the linear operator [24]

$$\hat{L}_{\mathrm{FP}}\rho=-\nabla\cdot(\mathbf{F}\rho)+D\Delta\rho \ . \tag{4.4}$$

We note that the Liouville operator is recovered in the limit $D\to 0$. As a consequence, we may wonder whether there is a close relationship between the eigenvalue problems of the deterministic and stochastic systems. The purpose of this section is to show that it is indeed the case. This will lead us to a better understanding of the meaning of the spectral decomposition of the deterministic Liouville equation and, in particular, of the status of the generalized functions (distributions) encountered in this decomposition.

Before we begin the analysis we recall that, for the one-dimensional systems we are concerned with, the vector field can be obtained as the gradient of a potential U according to

$$F(x)=-\partial_x U \ . \tag{4.5}$$

The function $\rho=\mathcal{N}_0\exp(-U/D)$ is known to be the stationary solution of the Fokker-Planck equation corresponding to the eigenvalue $\sigma_0=0$ ($\mathcal{N}_0$ is a normalization constant). The Fokker-Planck equation (4.3) can then be transformed into a Schrödinger-like form by setting

$$\rho=\exp\left[-\frac{U}{2D}\right]f \ . \tag{4.6}$$

The equation for the function f becomes

$$\partial_t f=-\hat{H}f \ , \tag{4.7}$$

where we introduced the Hamiltonian-like operator

$$\hat{H}=-D\partial_x^2+\Phi \ , \tag{4.8}$$

with the function Φ playing the role of a potential in the Schrödinger equation

$$\Phi=-\tfrac{1}{2}\partial_x^2 U+\frac{1}{4D}(\partial_x U)^2 \ . \tag{4.9}$$

The operator $\hat{H}$ is self-adjoint contrary to the Fokker-Planck operator, whose adjoint is $\hat{L}_{\mathrm{FP}}^{\dagger}=D\Delta+\mathbf{F}\cdot\nabla$. Typically, the "potential" Φ is indefinitely growing at large distances $x\to\pm\infty$ so that the eigenvalues of $\hat{H}$ are discrete. Solving the eigenvalue problem of $\hat{H}$,

$$\hat{H}\psi_m=-\sigma_m\psi_m \ , \tag{4.10}$$

we obtain the right and left eigenfunctions of the Fokker-Planck equation

$$\hat{L}_{\mathrm{FP}}\varphi_m=\sigma_m\varphi_m \quad \text{with } \varphi_m=\exp\left[-\frac{U}{2D}\right]\psi_m \ , \tag{4.11}$$

$$\hat{L}_{\mathrm{FP}}^{\dagger}\tilde{\varphi}_m=\sigma_m\tilde{\varphi}_m \quad \text{with } \tilde{\varphi}_m=\exp\left[+\frac{U}{2D}\right]\psi_m \ , \tag{4.12}$$

in terms of which we have the same spectral decomposition as in the previous sections with the biorthnormality relation $\langle\tilde{\varphi}_m,\varphi_n\rangle=\delta_{mn}$

$$\rho(x,t)=\sum_{m=0}^{\infty}\langle\tilde{\varphi}_m,\rho_0\rangle\exp(\sigma_m t)\varphi_m(x) \ . \tag{4.13}$$

We now consider, successively, the linear and the pitchfork bifurcation model analyzed already in the previous sections.

A. The case $\dot{x}=\mu x$

For this linear case, the vector field is $F(x)=\mu x$, while the corresponding potential is $U=-\mu x^2/2$. The Schrödinger-type potential is harmonic, given by

$$\Phi=\frac{\mu^2}{4D}x^2+\frac{\mu}{2}\ , \tag{4.14}$$

so that the eigenvalues are

$$\sigma_m=-|\mu|(m+1)\quad \text{for } \mu>0\ , \tag{4.15}$$

$$\sigma_m=-|\mu|m\quad \text{for } \mu<0\ , \tag{4.16}$$

with $m=0,1,2,3,\ldots$. They precisely coincide with the eigenvalues of the Liouville equation in this special linear case: $\sigma_n=s_n$. However, here the right and left eigenvectors are functions involving Hermite polynomials rather than distributions: for $\mu<0$,

$$\varphi_m=\mathcal{N}_m H_m(\alpha x)\exp\left[-\frac{|\mu|}{2D}x^2\right]\ ,$$
$$\tilde{\varphi}_m=\tilde{\mathcal{N}}_m H_m(\alpha x)\ , \tag{4.17}$$

and for $\mu>0$

$$\varphi_m=\tilde{\mathcal{N}}_m H_m(\alpha x)\ ,$$
$$\tilde{\varphi}_m=\mathcal{N}_m H_m(\alpha x)\exp\left[-\frac{|\mu|}{2D}x^2\right]\ , \tag{4.18}$$

with $\alpha=\sqrt{|\mu|/(2D)}$ and the Hermite polynomials

$$H_0(\xi)=1,\quad H_1(\xi)=2\xi,\quad H_2(\xi)=4\xi^2-2,\ldots\ . \tag{4.19}$$

We can now understand what happens in the limit $D\to 0$ or $\alpha\to\infty$. Let us consider the attracting case $\mu<0$. The right eigenfunctions φ_n are products of Hermite polynomials with a Gaussian of vanishing width (proportional to $\sqrt{D}\to 0$). The effect of Hermite polynomials is to introduce nodes in this peaked function so that the eigenfunction has successive maxima and minima, all occurring in a small domain of width $\sqrt{D}$ around the fixed point $x=0$. The integral of φ_m with an arbitrary function would give asymptotically ($D\to 0$) the mth derivative of the function at $x=0$. Let us apply this reasoning case by case. Since the first Hermite polynomial is constant, the eigenfunction φ_0 converges to the Dirac distribution $\delta(x)$ as $D\to 0$ up to an appropriate constant. This result is expected from the term $n=0$ of the deterministic spectral decomposition (2.18). Similarly, the next eigenfunction φ_1 has a minimum and a maximum on each side of $x=0$. Considered as a distribution, φ_1 is converging to the first derivative $-\delta^{(1)}(x)$ of the Dirac distribution as $D\to 0$. This reasoning can be developed systematically thanks to the Rodrigues formula according to which [22]

$$H_m(\xi)=(-1)^m\exp(\xi^2)\frac{d^m}{d\xi^m}\exp(-\xi^2)\ . \tag{4.20}$$

Since the Dirac distribution is the limit of a Gaussian function of arbitrarily small width, we obtain

$$\lim_{D\to 0}\varphi_m(x;D)=(-1)^m\delta^{(m)}(x)\quad (\mu<0) \tag{4.21}$$

if the renormalization constant is fixed to $\mathcal{N}_m=\alpha^{m+1}/\sqrt{\pi}$ in (4.17). Equation (4.21) is the result obtained directly by solving the Liouville equation.

Conversely, the left eigenfunctions $\tilde{\varphi}_m$ in (4.17) are polynomials of degree m. Since $H_m(\xi)=(2\xi)^m+O(\xi^{m-1})$, the monomial of higher degree dominates the other monomials when $\alpha\to\infty$. As a consequence, we obtain the left eigenvectors of the Liouville equation

$$\lim_{D\to 0}\tilde{\varphi}_m(x;D)=\frac{x^m}{m!}\quad (\mu<0)\ , \tag{4.22}$$

with the normalization constant $\tilde{\mathcal{N}}_m=[(2\alpha)^m m!]^{-1}$.

In conclusion, we recover the spectral decomposition of the Liouville equation by taking the limit $D\to 0$ of the spectral decomposition of the corresponding Fokker-Planck equation. Let us remark that, before taking this limit, one needs to modify the functional space $\mathcal{L}^2(\mathbb{R})$ of the Fokker-Planck spectral decomposition because the resulting distributions are not defined on $\mathcal{L}^2$ spaces. Similar considerations hold for the case $\mu>0$.

B. The case $\dot{x}=\mu x-x^3$

The Langevin equation reads

$$\dot{x}=\mu x-x^3+W(t) \tag{4.23}$$

and its deterministic part undergoes a pitchfork bifurcation at $\mu=0$. The vector-field potential is

$$U=-\frac{\mu}{2}x^2+\frac{x^4}{4} \tag{4.24}$$

and the Schrödinger-type potential is

$$\Phi=\frac{x^2}{4D}(x^2-\mu)^2+\frac{\mu}{2}-\frac{3}{2}x^2\ . \tag{4.25}$$

This potential is depicted in Fig. 2. When $\mu>0$, the three minima tend to the same level as $D\to 0$, so that we may expect quasiharmonic behavior in the three wells, which correspond to the three fixed points: the two stable ones $x=\pm\sqrt{\mu}$ and the unstable one $x=0$. On the other hand, there is only one well when $\mu<0$. The rescaling

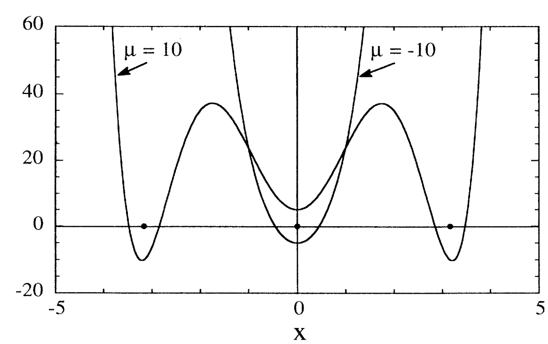

FIG. 2. Schrödinger-type potential (4.25) for the Fokker-Planck equation of the pitchfork bifurcation for two values of the parameter μ below and above criticality.

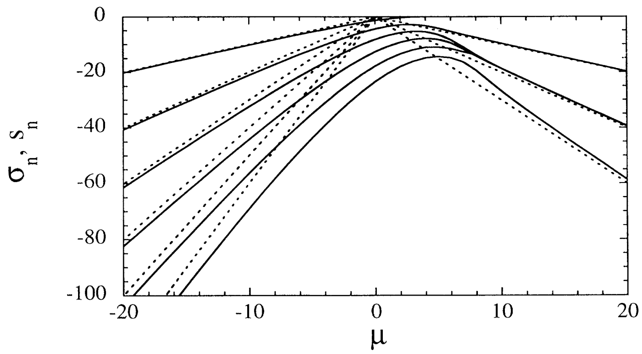

FIG. 3. Eigenvalues σ_m of the Fokker-Planck operator (4.4) of the pitchfork bifurcation versus the bifurcation parameter μ. The eigenvalues σ_m are compared with the eigenvalues $s_n=-n|\mu|$ of the deterministic Liouville operator.

$$x \to D^{1/4}x, \quad t \to D^{-1/2}t, \quad \mu \to D^{1/2}\mu, \quad \sigma_m \to D^{1/2}\sigma_m \tag{4.26}$$

shows that the diffusion coefficient can equivalently be set equal to unity ($D=1$) and that the limit $D \to 0$ is equivalent to the limit $|\mu| \to \infty$.

We have numerically integrated the eigenvalue equation (4.10) to obtain the eigenvalues σ_m and the corresponding eigenfunctions as a function of the parameter μ through the pitchfork bifurcation (see Fig. 3). For large and negative values of μ before the pitchfork bifurcation, $x=0$ is attracting in the deterministic system with a stability exponent $\lambda=-|\mu|$, so that the problem is essentially identical to the case $\mu<0$ of the linear model. Indeed, all the eigenvalues are well separated and we see that they converge to the simply degenerate eigenvalues $s_n=-|\mu|n$ of the Liouville equation. This behavior is confirmed by

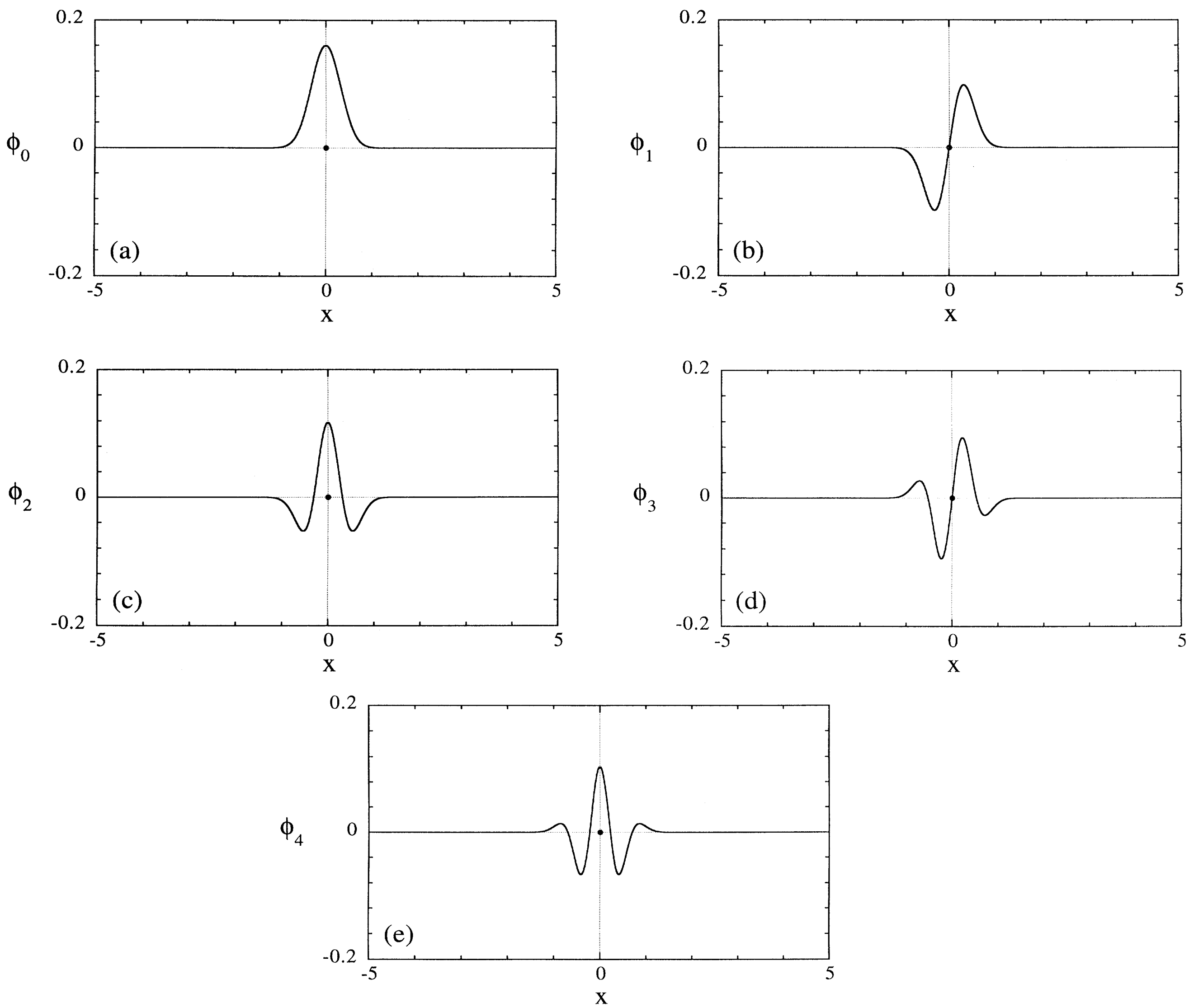

FIG. 4. (a)–(e) Right eigenfunctions (4.11) associated with the five lowest eigenvalues of the Fokker-Planck operator below the pitchfork bifurcation $\mu<0$. The dot marks the location of the single attracting fixed point.

inspection of the eigenfunctions in Fig. 4, which resemble the eigenfunctions of the harmonic oscillator.

The situation is different beyond the bifurcation at $\mu>0$. We observe that the eigenvalues again converge toward the Liouville eigenvalues as $\mu\to+\infty$. In this limit, we observe a clustering in the way predicted by the degeneracies g_n of the deterministic eigenvalues, which are

$$s_n=0,-\mu,-2\mu,-3\mu,-4\mu,-5\mu,\ldots,$$
$$g_n=2,1,3,1,3,1,\ldots. \tag{4.27}$$

Indeed, the deterministic eigenvalue $s_0=0$ corresponds to two independent Dirac distributions centered respectively on the two attracting fixed points $x=\pm\sqrt{\mu}$. In the stochastic system, we have the stationary state of even parity

$$\varphi_0=\mathcal{N}_0\exp\left[-\frac{x^4}{4D}+\frac{\mu}{2D}x^2\right], \tag{4.28}$$

with the eigenvalue $\sigma_0=0$. The left eigenfunction is $\tilde{\varphi}_0=1$. We also find another eigenvalue, which is exponentially close to $\sigma_0=0$ [$\sigma_1\sim-\exp(-D^{-1})$], which corresponds to a right eigenfunction of odd parity shown in Fig. 5 [25]. For small D, these right eigenfunctions reduce to

$$\varphi_0\sim e^{-(\mu/D)(x-\sqrt{\mu})^2}+e^{-(\mu/D)(x+\sqrt{\mu})^2}, \tag{4.29}$$

$$\varphi_1\sim e^{-(\mu/D)(x-\sqrt{\mu})^2}-e^{-(\mu/D)(x+\sqrt{\mu})^2}. \tag{4.30}$$

The overlap between the exponential tails explains the exponentially small separation between the eigenvalues. In

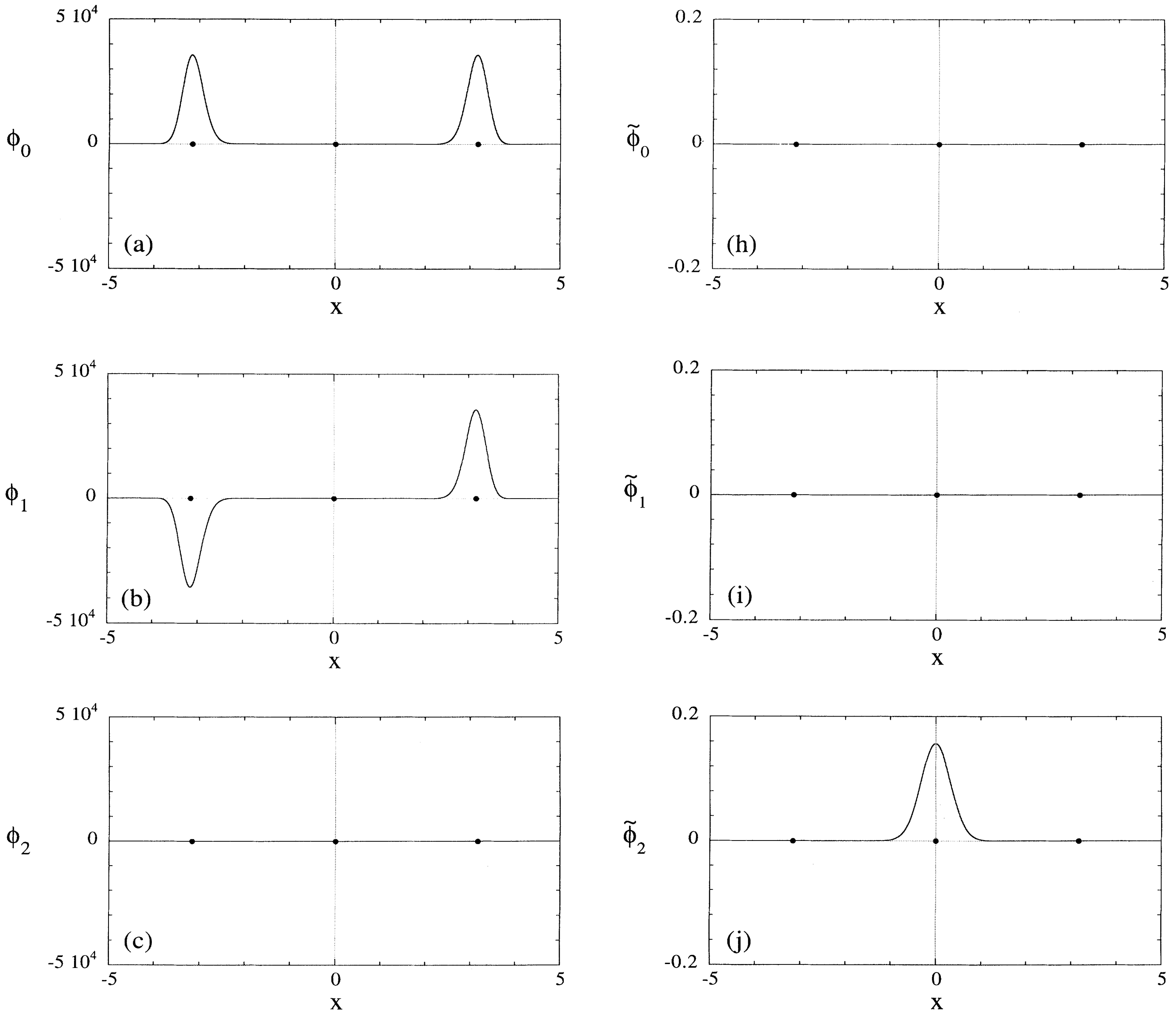

FIG. 5. (a)–(g) Right and (h)–(n) left eigenfunctions (4.11) and (4.12) associated with the seven lowest eigenvalues of the Fokker-Planck operator beyond the pitchfork bifurcation $\mu>0$. The dots mark the locations of the fixed points.

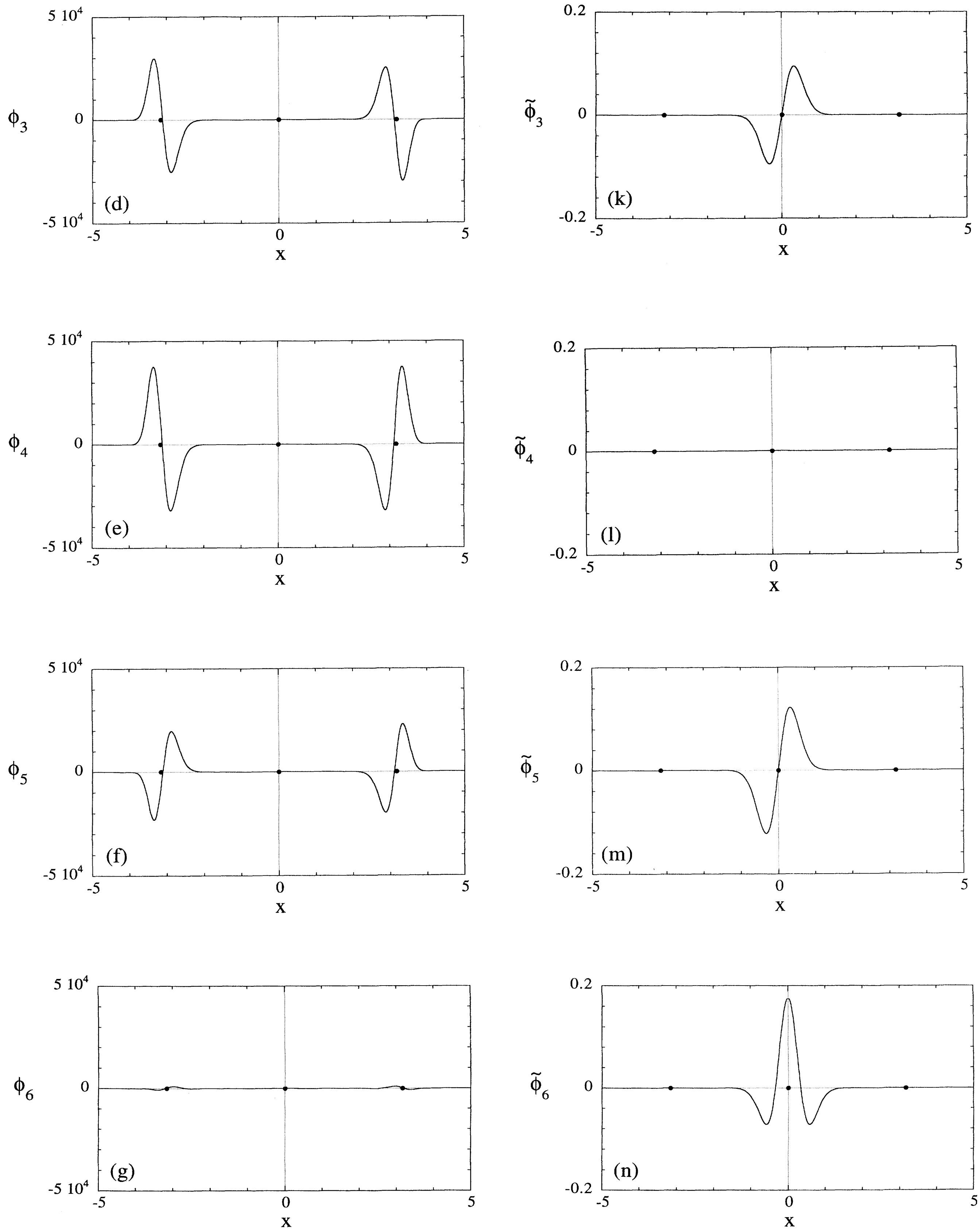

FIG. 5. (*Continued*).

the limit $D \to 0$, one obtains the distributions

$$\varphi_0 \underset{D\to 0}{\sim} \delta(x-\sqrt{\mu})+\delta(x+\sqrt{\mu}) \ , \tag{4.31}$$

$$\varphi_1 \underset{D\to 0}{\sim} \delta(x-\sqrt{\mu})-\delta(x+\sqrt{\mu}) \ , \tag{4.32}$$

which are linear combinations of even and odd parities of Dirac distributions found for the Liouville equation.

The next right eigenfunction of σ_2 is of even parity [23]. The left eigenfunction tends to a Dirac distribution centered with the unstable fixed point $x=0$ (see Fig. 5). The corresponding eigenvalue $s_1=-\mu$ of the deterministic Liouville equation is the first eigenvalue associated with the unstable fixed point. The fact that we recover this eigenvalue in the Fokker-Planck equation shows that it is intrinsically associated with the problem and that it is not an artifact of the method used in the previous sections.

Thereafter, a cluster of three exponentially separated eigenvalues, corresponding to the triply degenerate eigenvalue -2μ of the Liouville equation, appears

$$\lim_{D\to 0} \sigma_3,\sigma_4,\sigma_5=-2\mu \ . \tag{4.33}$$

In the deterministic system, two of these eigenvalues have their right eigenvectors composed by first derivatives of the Dirac distribution centered on the two stable fixed points $x=\pm\sqrt{\mu}$ and the left eigenvector of the remaining eigenvalue is a first derivative of Dirac distribution, but now centered on the unstable fixed point $x=0$. This is confirmed by inspection of the right and left eigenfunctions of σ_3, σ_4, and σ_5 shown in Fig. 5. The right eigenfunction φ_4 converges to the even-parity combination $\delta^{(1)}(x-\sqrt{\mu})-\delta^{(1)}(x+\sqrt{\mu})$. On the other hand, we observe in Fig. 5 that a weighted difference of the odd-parity right eigenfunctions φ_3 and φ_5 can be constructed, which would converge to $\delta^{(1)}(x-\sqrt{\mu})+\delta^{(1)}(x+\sqrt{\mu})$, while the corresponding linear combination of the left eigenfunctions $\tilde{\varphi}_3$ and $\tilde{\varphi}_5$ would have a vanishing distributionlike component when $D\to 0$. Conversely, a weighted sum of the right eigenfunctions φ_3 and φ_5 can be constructed that has a vanishing distributionlike component when $D\to 0$, although the same linear combination of the *left eigenfunctions* $\tilde{\varphi}_3$ and $\tilde{\varphi}_5$ would converge to the distribution $\delta^{(1)}(x)$. In this way, we can establish the correspondence between the spectral decompositions of the stochastic and deterministic equations.

At the bifurcation $\mu=0$, the continuous spectrum of the Liouville equation, which is the signature of the critical slowing down, is transformed into a discrete spectrum at the level of the Fokker-Planck equation. In this region, the Liouville spectrum is of little use to understand the eigenvalues and eigenfunctions of the Fokker-Planck equation. However, in the limit $D\to 0$, the range of this region shrinks to zero and the eigenvalues accumulate at $\mu=0$ so that we may understand why the deterministic system has the continuous spectrum at $\mu=0$ described in Sec. III.

C. A correspondence theorem

We have seen that, far from bifurcation, there is an interesting correspondence between the eigenvalue problems of the stochastic and deterministic systems, the latter being in the noiseless limit ($D\to 0$) of the former. This correspondence can be rendered more precise with the following theorem.

Let the one-dimensional vector field $\dot{x}=F(x)$ possess a countable set of separated fixed points $\{x_i\}$, $F(x_i)=0$, without accumulation point and be such that none of the stability exponents is vanishing

$$\lambda_i=\frac{dF}{dx}(x_i)\neq 0 \ . \tag{4.34}$$

The spectrum of the Fokker-Planck equation (4.3) converges then to the spectrum of the Liouville equation in the noiseless limit $D\to 0$. Appropriate linear combinations of the right and left eigenfunctions of the Fokker-Planck equation converge in the sense of distribution to the right and left eigenvectors of the Liouville equation if there is no Jordan-block structure.

We can prove this theorem by noting that the Schrödinger-type potential

$$\Phi=\frac{F(x)^2}{4D}-\frac{1}{2}\partial_x F(x) \tag{4.35}$$

is dominated by the first term as $D\to 0$. Figure 6 schematically shows such a typical potential, formed by a sequence of wells with minima at the fixed points. The barriers between the wells grow when $D\to 0$. Near the fixed points the potential is approximately harmonic

$$\Phi\simeq\frac{\lambda_i^2}{4D}(x-x_i)^2-\frac{\lambda_i}{2} \tag{4.36}$$

since $F(x)\simeq\lambda_i(x-x_i)$. As a consequence, the behavior of the eigenvalues and eigenfunctions is that of a set of harmonic oscillators near the bottom of the wells. As $D\to 0$, more and more of them join the set of harmonic-like eigenvalues because the barriers grow. Therefore, each well tends to become independent of its neighbors

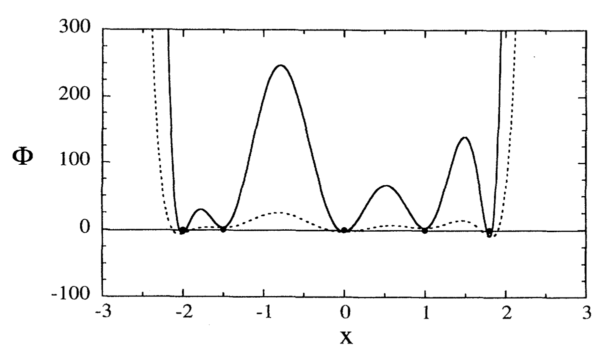

FIG. 6. Typical Schrödinger-type potential (4.35) of a 1D vector field with five fixed points. The dashed line is the potential for a relatively large value of the diffusion coefficient D, while the solid line is the same, but for a relatively smaller value of the diffusion coefficient, which shows the deepening of the wells separating the fixed points.

up to corrections which are exponentially small like $\exp(-D^{-1})$ [25]. The eigenvalues of the Fokker-Planck equation are given in terms of the independent harmonic oscillators

$$\lim_{D\to 0} \sigma_{n_i}^{(i)}(D) = -|\lambda_i| n_i + \frac{\lambda_i - |\lambda_i|}{2} \tag{4.37}$$

and are precisely the eigenvalues of the Liouville equation. Concerning the right and left eigenfunctions, the preceding discussion shows that they tend to behave like linear combinations of the results (4.17) and (4.18) locally around each fixed point x_i. In the limit $D\to 0$, we already proved that (4.17) and (4.18) converge to either the functions or the distributions appearing in the spectral decomposition of the Liouville equation in the limit $D\to 0$ when there is no Jordan-block structure. Therefore, the same is true for the corresponding linear combinations, which ends the proof of the theorem. When the Liouville equation presents a Jordan-block structure as in Sec. IV B, a further analysis is required to establish a correspondence between the eigenfunctions of the Fokker-Planck operator and the eigenvectors and other radical vectors of the Liouville operator, but we expect that the correspondence extends to the general case also under appropriate conditions. This discussion of the correspondence between the Fokker-Planck and the Liouville equations sheds further light on our approach to the solution of the Liouville equation.

V. CONCLUSIONS

In this paper we constructed explicitly the spectral decompositions of the Liouvillian dynamics of nonlinear vector fields possessing fixed points. The Liouvillian dynamics is commonly encountered in every statistical treatment of dynamical systems so that these spectral decompositions are governing not only time averages of observables, but also dynamic properties such as multiple-time correlation functions and associated power spectra. We have shown that a special kind of mathematics is necessary to describe the spectral decomposition of the Liouvillian dynamics of deterministic systems which involves distributions rather than regular functions used in noisy systems.

We have essentially limited the analysis to one-dimensional vector fields, especially a canonical model of the pitchfork bifurcation. We showed that the spectrum is discrete below and above criticality, but becomes continuous at criticality, reflecting the critical slowing down associated with bifurcation. Above criticality, the spectral decomposition of the Liouville operator shows a Jordan-block structure as a result of degeneracies in the spectrum. This is the signature, at the spectral level, of the phenomenon of symmetry breaking induced by the pitchfork bifurcation.

We established a correspondence theorem between the spectral decompositions of the Liouvillian operator governing the deterministic dynamics and the Fokker-Planck operator of the associated stochastic dynamics. In particular, the eigenvalues of the stochastic problem converge to those of the deterministic dynamics in the noiseless limit if the system is far from criticality. On the other hand, our results show that a discrete spectrum of the Fokker-Planck operator may turn into a continuous spectrum for the corresponding Liouville operator if the system is at criticality. Our results are very general and can be extended to other types of bifurcations involving fixed points such as, in particular, the tangent bifurcation and the Hopf bifurcation.

ACKNOWLEDGMENTS

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APPENDIX A: RELATION BETWEEN THE FUNCTIONAL SPACES (2.21) AND (2.22)

In this appendix we prove that the choice of the observable functional space (2.22) is the necessary and sufficient condition so that the series (2.18) is well defined in the sense of distribution once the density functional space $\mathcal{F}_\rho$ is fixed as in (2.21). Indeed, if $\rho_0(x)\in\mathcal{C}_0^\infty(\mathbb{R})$, there exist positive constants C and k such that the coefficients of the series (2.18) are bounded by $|\int_{-\infty}^{+\infty} y^n \rho_0(y)dy| \le Ck^n$. Therefore, the averages $\langle A,\rho_t\rangle$ obtained by applying (2.18) on an observable $A(x)\in\mathcal{E}$ are bounded by

$$|\langle A,\rho_t\rangle| \le C \sum_{n=0}^{\infty} \frac{|A^{(n)}(0)|}{n!}(e^{\mu t}k)^n , \tag{A1}$$

which converges for all times $t\ge 0$ because A is an entire function, so that the spectral decomposition is well defined.

Conversely, let us suppose that the average $\langle A,\rho_t\rangle$ obtained from (2.18) converges absolutely for all $\rho_0(x)\in\mathcal{C}_0^\infty(\mathbb{R})$ and for all times $t\ge 0$, i.e.,

$$\sum_{n=0}^{\infty} e^{n\mu t}\frac{|A^{(n)}(0)|}{n!} \left| \int_{-\infty}^{+\infty} y^n \rho_0(y)dy \right| < \infty . \tag{A2}$$

For an arbitrary positive number $R>0$, we can find a function $\rho_0(x)\in\mathcal{C}_0^\infty(\mathbb{R})$ such that (a) $\rho_0(x)\ge 0$, (b) $\rho_0(x)=0$ for $x<0$, and (c) $\rho_0(x)=1$ for $R\le x\le 2R$. For this function ρ_0, the coefficients of (2.18) are bounded from below according to $|\int_{-\infty}^{+\infty} y^n\rho_0(y)dy| \ge R^{n+1}$. Therefore, we obtain

$$\sum_{n=0}^{\infty} \frac{|A^{(n)}(0)|}{n!} R^{n+1} \leq \sum_{n=0}^{\infty} \frac{|A^{(n)}(0)|}{n!} \left| \int_{-\infty}^{+\infty} y^n \rho_0(y) dy \right| < \infty , \quad \text{(A3)}$$

which implies the absolute convergence of the Taylor series of $A(x)$ and therefore its analyticity for $|x| < R$. Since R is arbitrary, we finally have that $A \in \mathcal{E}$. Q.E.D.

We remark that, if $A(x) \in \mathcal{C}^{\infty}(\mathbb{R})$, which is the set of infinitely differentiable functions with arbitrary support, for instance, such as $A(x) = (x^2+1)^{-1}$, the average $\langle A, \rho_t \rangle$ may diverge for certain initial densities ρ_0 when time is positive but small enough. For these reasons, we have to adopt the choice (2.22).

APPENDIX B: RELATION OF $\phi_{s\pm}$ TO χ_s [EQ. (3.35)] IN THE CRITICAL REGIME $\mu = 0$

Our purpose is to show how the right eigendistributions of the critical Liouville operator are related to the eigenfunction (3.35). We proceed by transforming the Gamma function appearing in (3.25) into a contour integral using Hankel's formula [22]

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C dt (-t)^{-z} \exp(-t) , \quad \text{(B1)}$$

where C is the contour of Fig. 7.

Multiplying (B1) by a monomial in s and changing the integration variable to $x = [(-s)/(-2t)]^{1/2}$, we see that the corresponding parts of the distributions (3.29) and (3.31) are transformed into

$$\frac{1}{2\Gamma(n/2)} \left[-\frac{s}{2} \right]^{n/2-1} = \frac{i}{2\pi} \int_\Gamma dx\, x^{n-3} \exp\left[-\frac{s}{2x^2} \right] , \quad \text{(B2)}$$

where Γ is the contour of Fig. 8(a). Thanks to this integral representation of the individual terms entering the distributions (3.29) and (3.31), we can express the action of these distributions on an observable $A(x)$ as the contour integrals

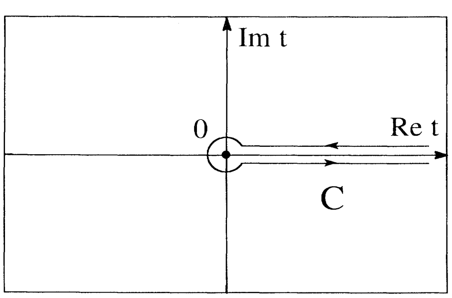

FIG. 7. Contour C entering in the Hankel formula (B1).

$$\int_{-\infty}^{+\infty} dx\, \phi_{s+}(x) A(x) = \frac{i}{2\pi} \int_\Gamma \frac{dx}{x^3} \exp\left[-\frac{s}{2x^2} \right] \times \left[\frac{A(x)+A(-x)}{2} - A(0) \right] , \quad \text{(B3)}$$

$$\int_{-\infty}^{+\infty} dx\, \phi_{s-}(x) A(x) = \frac{i}{2\pi} \int_\Gamma \frac{dx}{x^3} \exp\left[-\frac{s}{2x^2} \right] \frac{A(x)-A(-x)}{2} . \quad \text{(B4)}$$

We recognize in these formulas the eigenfunction (3.35). The even-parity formula (B3) still contains an extra term $A(0)$ on its right-hand side which is due to the discrete eigenvalue $s=0$ rather than to the continuous spectrum. In order to remove this term, another contour Γ' shown in Fig. 8(b) is used in terms of which both the even- and odd-parity sectors are handled similarly

$$\int_{-\infty}^{+\infty} dx\, \phi_{s\pm}(x) A(x) = \frac{i}{2\pi} \int_{\Gamma'} \frac{dx}{x^3} \exp\left[-\frac{s}{2x^2} \right] \frac{A(x) \pm A(-x)}{2} . \quad \text{(B5)}$$

The relation of the right eigendistributions (3.29) and (3.31) to the eigenfunction (3.35) is now straightforward.

To prove the equivalence of (B5) with (B3) and (B4), we multiply (B5) by $\exp[s/(2x_0^2)]$ and we integrate over $\int_{-\infty}^{0} ds$. Because of

$$\int_{-\infty}^{0} ds \exp\left[\frac{s}{2x_0^2} - \frac{s}{2x^2} \right] = \frac{2x_0^2 x^2}{x^2 - x_0^2} , \quad \text{(B6)}$$

we get

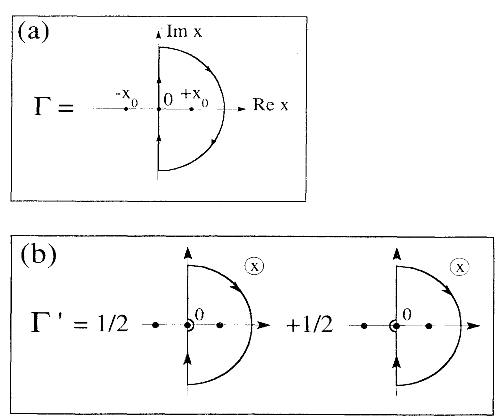

FIG. 8. (a) Contour Γ entering in Eqs. (B3) and (B4). (b) Contour Γ' used in Eqs. (B5) and (B7).

$$\int_{-\infty}^{0} ds \exp\left[\frac{s}{2x_0^2}\right]\left[\int dx\, \phi_{s\pm}(x)A(x)\right]=\frac{i}{2\pi}\int_{\Gamma'} dx \frac{A(x)\pm A(-x)}{2}\left[\frac{1}{x-|x_0|}+\frac{1}{x+|x_0|}-\frac{2}{x}\right]$$
$$=\begin{cases}\dfrac{A(x)+A(-x)}{2}-A(0) & \text{(even parity)}\\[2ex] \dfrac{A(x)-A(-x)}{2} & \text{(odd parity)}\end{cases} \tag{B7}$$

This result is obtained because the integration with the contour Γ' gives a full contribution at $x=|x_0|$ plus a half contribution at the origin $x=0$ (see Fig. 8). The result (B7) is identical to the result we would have obtained using (B3) and (B4) instead of (B5). Thanks to (B7), we also proved as a corollary the completeness of the spectral decomposition

$$A(x)=A(0)+\sum_{\rho=\pm}\int_{-\infty}^{0} ds\langle A,\phi_{s\rho}\rangle\tilde{\phi}_{s\rho}(x) . \tag{B8}$$

APPENDIX C: CONSTRUCTION OF THE SPECTRAL DECOMPOSITION IN THE SUPERCRITICAL REGIME $\mu>0$

Our purpose here is to explicitly construct the first three invariant (radical) subspaces of the spectral decomposition (3.49) in the internal interval $(0,+\sqrt{\mu})$, denoted as $1+$. We start from the phase-space averages in either the final position x or the initial position $y=x_0$, given by Eqs. (3.47) and (3.48). Taking the limit $t\to+\infty$ in (3.47), we get

$$\langle A\rangle_t \to A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy , \tag{C1}$$

from which we immediately obtain the first eigenspace associated with the eigenvalue $s_0=0$ [(3.50) and (3.51)]. This eigenspace gives us the invariant probability density.

Subtracting this asymptotic behavior from the full average and changing the integration variable from y to x we get, using (3.48),

$$\langle A\rangle_t - A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy = \int_0^{\sqrt{\mu}} dx \frac{\mu^{3/2}\exp(-\mu t)}{[\mu-x^2+x^2\exp(-2\mu t)]^{3/2}} \times[A(x)-A(\sqrt{\mu})] \times\rho_0\left[\frac{x\sqrt{\mu}\exp(-\mu t)}{\sqrt{\mu-x^2+x^2\exp(-2\mu t)}}\right] . \tag{C2}$$

In the limit $t\to\infty$, Eq. (C2) leads to

$$\exp(\mu t)\left[\langle A\rangle_t - A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy\right] \to\rho_0(0)\int_0^{\sqrt{\mu}}\frac{\mu^{3/2}}{(\mu-x^2)^{3/2}}[A(x)-A(\sqrt{\mu})] , \tag{C3}$$

so that we can identify the second eigenspace associated with $s_1=-\mu$ [(3.52) and (3.53)].

Subtracting now the first two terms from the term average, we get

$$\begin{aligned}\langle A\rangle_t - A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy-\exp(-\mu t)\rho_0(0)\int_0^{\sqrt{\mu}}\frac{\mu^{3/2}}{(\mu-x^2)^{3/2}}[A(x)-A(\sqrt{\mu})]\\ =\int_0^{\sqrt{\mu}}dx\frac{\mu^{3/2}\exp(-\mu t)}{[\mu-x^2+x^2\exp(-2\mu t)]^{3/2}}[A(x)-A(\sqrt{\mu})]\\ \times\left\{\rho_0\left[\frac{x\sqrt{\mu}\exp(-\mu t)}{\sqrt{\mu-x^2+x^2\exp(-2\mu t)}}\right]-\left[\frac{\mu-x^2+x^2\exp(-2\mu t)}{\mu-x^2}\right]^{3/2}\rho_0(0)\right\}\end{aligned} \tag{C4}$$

$$=\int_0^{\sqrt{\mu}}dy\left\{A\left[\frac{y\sqrt{\mu}}{\sqrt{y^2+(\mu-y^2)\exp(-2\mu t)}}\right]-A(\sqrt{\mu})\right\}\left[\rho_0(y)-\left[\frac{\mu}{\mu-y^2}\right]^{3/2}\rho_0(0)\right] . \tag{C5}$$

In (C5), the behavior of A around $y=\sqrt{\mu}$ is given by

$$A\left[\frac{y\sqrt{\mu}}{\sqrt{y^2+(\mu-y^2)\exp(-2\mu t)}}\right]-A(\sqrt{\mu})=A'(\sqrt{\mu})\frac{y^2-\mu}{2\sqrt{\mu}}\exp(-2\mu t)+O((y-\sqrt{\mu})^2) . \tag{C6}$$

On the other hand, the function ρ_0 appearing in (C4) is expanded in a Taylor series of the variable $[x\sqrt{\mu}\exp(-\mu t)/\sqrt{\mu-x^2}]$ and we keep the leading term of the series given by

$$\rho_0\left[\frac{x\sqrt{\mu}\exp(-\mu t)}{\sqrt{\mu-x^2+x^2\exp(-2\mu t)}}\right]-\left[\frac{\mu-x^2+x^2\exp(-2\mu t)}{\mu-x^2}\right]^{3/2}\rho_0(0) =\rho_0'(0)\frac{x\sqrt{\mu}}{\sqrt{\mu-x^2}}\exp(-\mu t)+O(x^2\exp(-2\mu t)/(\mu-x^2)) . \tag{C7}$$

From Eq. (C6), we infer that the term we need to subtract from the bracket containing $A(x)$ in (C4) is $A'(\sqrt{\mu})(x^2-\mu)/(2\sqrt{\mu})$. The subtracted term must then be added as a remaining term. Accordingly, Eq. (C4) is split in two different parts

$$\langle A\rangle_t - A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy - \exp(-\mu t)\rho_0(0)\int_0^{\sqrt{\mu}}\frac{\mu^{3/2}}{(\mu-x^2)^{3/2}}[A(x)-A(\sqrt{\mu})]$$
$$= \int_0^{\sqrt{\mu}} dx\frac{\mu^{3/2}\exp(-\mu t)}{[\mu-x^2+x^2\exp(-2\mu t)]^{3/2}}\left[A(x)-A(\sqrt{\mu})-\frac{x^2-\mu}{2\sqrt{\mu}}A'(\sqrt{\mu})\right]$$
$$\times\left\{\rho_0'(0)\frac{x\sqrt{\mu}}{\sqrt{\mu-x^2}}\exp(-\mu t)+O\left[\frac{x^2\exp(-2\mu t)}{\mu-x^2}\right]\right\}$$
$$-\frac{\sqrt{\mu}}{2}A'(\sqrt{\mu})\exp(-2\mu t)\int_0^{\sqrt{\mu}}dy\frac{\mu-y^2}{y^2+(\mu-y^2)\exp(-2\mu t)}\left[\rho_0(y)-\left[\frac{\mu}{\mu-y^2}\right]^{3/2}\rho_0(0)\right], \tag{C8}$$

where we used (C7) as well as a change of integration to $y=x_0$ in the last term, as done in (C5). As a consequence of the change of variable, $x^2-\mu$ has been transformed into the particular function of y^2 and $\exp(-2\mu t)$ appearing in the last integral of (C8).

Let us discuss the two terms composing (C8). The integral of the first term is convergent near $x=0$ and $x=\sqrt{\mu}$ since the bracket involving A behaves like $(x-\sqrt{\mu})^2$ near $x=\sqrt{\mu}$. This suffices to compensate for the singularity which appears in the denominator in the limit $t\to+\infty$, so that the integral is finite. As a consequence, we can perform the limit $t\to+\infty$ in this first term and identify a meaningful coefficient to $\exp(-2\mu t)$.

However, the second term is problematic near $y=0$ since the bracket with ρ_0 behaves like y, while the denominator behaves like y^2 after the limit $t\to+\infty$ is taken so that this integral is divergent. To avoid this divergent integral, we add and subtract a term behaving like $y\rho_0'(0)$ in this bracket and get

$$\langle A\rangle_t - A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy - \exp(-\mu t)\rho_0(0)\int_0^{\sqrt{\mu}}\frac{\mu^{3/2}}{(\mu-x^2)^{3/2}}[A(x)-A(\sqrt{\mu})]$$
$$= \int_0^{\sqrt{\mu}} dx\frac{\mu^{3/2}\exp(-\mu t)}{[\mu-x^2+x^2\exp(-2\mu t)]^{3/2}}\left[A(x)-A(\sqrt{\mu})-\frac{x^2-\mu}{2\sqrt{\mu}}A'(\sqrt{\mu})\right]$$
$$\times\left\{\rho_0'(0)\frac{x\sqrt{\mu}}{\sqrt{\mu-x^2}}\exp(-\mu t)+O\left[\frac{x^2\exp(-2\mu t)}{\mu-x^2}\right]\right\}$$
$$-\frac{\sqrt{\mu}}{2}A'(\sqrt{\mu})\exp(-2\mu t)\int_0^{\sqrt{\mu}}dy\frac{\mu-y^2}{y^2+(\mu-y^2)\exp(-2\mu t)}$$
$$\times\left[\rho_0(y)-\left[\frac{\mu}{\mu-y^2}\right]^{3/2}\rho_0(0)-\frac{\mu y}{\mu-y^2}\rho_0'(0)\right]$$
$$-\frac{\mu^{3/2}}{2}A'(\sqrt{\mu})\rho_0'(0)\exp(-2\mu t)\int_0^{\sqrt{\mu}}dy\frac{\mu-y^2}{y^2+(\mu-y^2)\exp(-2\mu t)}\frac{y}{\mu-y^2}\ . \tag{C9}$$

The integral of the second term of (C9) is now convergent. The newly added third term is still problematic if we carry out the limit $t\to+\infty$ before the integral. However, a divergent integral does not arise if the integral is performed before the limit $t\to+\infty$ because we have that

$$\int_0^{\sqrt{\mu}}\frac{y}{y^2+(\mu-y^2)\exp(-2\mu t)}=\frac{\mu t}{1-\exp(-2\mu t)}\ . \tag{C10}$$

In this last term, therefore, an extra power of the time t, which is at the basis of the existence of a Jordan block, appears.

Finally, we get the asymptotic behavior

$$\langle A\rangle_t - A(\sqrt{\mu})\int_0^{\sqrt{\mu}}\rho_0(y)dy - \exp(-\mu t)\rho_0(0)\int_0^{\sqrt{\mu}}\frac{\mu^{3/2}}{(\mu-x^2)^{3/2}}[A(x)-A(\sqrt{\mu})]$$
$$=\exp(-2\mu t)\rho_0'(0)\int_0^{\sqrt{\mu}}\frac{\mu^2 x}{(\mu-x^2)^2}\left[A(x)-A(\sqrt{\mu})-\frac{x^2-\mu}{2\sqrt{\mu}}A'(\sqrt{\mu})\right]$$
$$-\exp(-2\mu t)\frac{\sqrt{\mu}}{2}A'(\sqrt{\mu})\int_0^{\sqrt{\mu}}dy\frac{\mu-y^2}{y^2}\left[\rho_0(y)-\left[\frac{\mu}{\mu-y^2}\right]^{3/2}\rho_0(0)-\frac{\mu y}{\mu-y^2}\rho_0'(0)\right]$$
$$-t\exp(-2\mu t)\frac{\mu^{5/2}}{2}A'(\sqrt{\mu})\rho_0'(0)+O(\exp(-3\mu t))+O(t\exp(-4\mu t))\ , \qquad \text{(C11)}$$

from which we can identify the third invariant subspace associated with $s_2=-2\mu$, given by (3.54)–(3.57). We have therefore obtained the spectral decomposition (3.49).

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[1] G. Nicolis and C. Nicolis, Phys. Rev. A **38**, 1 (1988).

[2] H. Hasegawa and W. Saphir, Phys. Lett. A **161**, 471 (1992); **161**, 477 (1992); W. C. Saphir and H. H. Hasegawa, *ibid.* **171**, 317 (1992); H. H. Hasegawa and D. J. Driebe, *ibid.* **176**, 193 (1993).

[3] P. Gaspard, J. Phys. A. **25**, L483 (1992); Phys. Lett. A **168**, 13 (1992).

[4] I. Antoniou and S. Tasaki, J. Phys. A. **26**, 73 (1992); S. Tasaki, I. Antoniou, and Z. Suchanecki, Phys. Lett. A **179**, 97 (1993); Chaos, Solitons, Fractals **4**, 227 (1994).

[5] P. Gaspard and D. Alonso Ramirez, Phys. Rev. A **45**, 8383 (1992).

[6] D. MacKernan and G. Nicolis, Phys. Rev. E **50**, 988 (1994).

[7] M. Malek Mansour, C. Van Den Broeck, G. Nicolis, and J. W. Turner, Ann. Phys. (N.Y.) **131**, 283 (1981).

[8] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[9] G. Nicolis and P. Gaspard, Chaos, Solitons, Fractals **4**, 41 (1994).

[10] G. Nicolis and I. Prigogine, *Exploring Complexity* (Freeman, New York 1989).

[11] I. Prigogine, *Nonequilibrium Statistical Mechanics* (Wiley, New York, 1962).

[12] N. Dunford and J. T. Schwartz, *Linear Operators* (Wiley Interscience, New York, 1958, 1963, 1971), Vols. I–III.

[13] The geometric multiplicity of an eigenvalue s_n is the dimension of the eigenspace associated with s_n, i.e., the dimension of the linear subspace of all vectors ϕ such that $(\hat{L}-s_n\hat{I})\phi=0$. The range of $\hat{L}$ is said to be closed if the set of vectors $\hat{L}\phi$ constitutes a proper subspace of the total functional space.

[14] B. Friedman, *Principles and Techniques of Applied Mathematics* (Wiley, New York, 1956).

[15] I. C. Gohberg and M. G. Krein, *Introduction à la Théorie des Opérateurs Linéaires non-Auto-Adjoint dans un Espace Hilbertien* (Dunod, Paris, 1971).

[16] G. Iooss and D. D. Joseph, *Elementary Stability and Bifurcation Theory* (Springer, New York, 1980).

[17] I. Gel'fand and G. Shilov, *Generalized Functions* (Academic, New York, 1968), Vol. 2.

[18] $A(x)$ is said to be an entire function if the Taylor expansion $A(x)=\sum_{n=0}^{\infty}A^{(n)}(0)x^n/n!$ converges for all real numbers x. We denote by $\mathscr{E}$ the set of entire functions.

[19] M. Pollicott, Invent. Math. **81**, 413 (1985); **85**, 147 (1986).

[20] D. Ruelle, Phys. Rev. Lett. **56**, 405 (1986); J. Stat. Phys. **44**, 281 (1986); J. Diff. Geom. **25**, 117 (1987).

[21] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer, New York, 1982).

[22] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. Stegun (Dover, New York, 1972).

[23] Note, that due to the degeneracies of the deterministic eigenvalues, the index of the Fokker-Planck eigenvalue σ_m differs from the index of the corresponding deterministic eigenvalue s_n. In particular, σ_0 and σ_1 correspond to s_0; σ_2 to s_1; $\sigma_3,\sigma_4,\sigma_5$ to s_2, etc.

[24] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1984).

[25] H. Dekker and N. G. van Kampen, Phys. Lett. **73A**, 374 (1979); N. G. van Kampen, J. Stat. Phys. **17**, 71 (1977); Suppl. Prog. Theor. Phys. **64**, 389 (1978).